Hamilton Cycles in Planar Graphs and Venn Diagrams

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Abstract: Using graph theory, we prove Grünbaum's conjecture [4]: Every Venn diagram of n curves can be extended to a Venn diagram of n+1 curves by the addition of a suitable simple closed curve.

1. Introduction: A Venn diagram consists of n simple closed curves in the plane so that all possible intersections (2^n many) of the interiors and the exteriors of these curves are nonempty and are connected. One can put various restrictions on the diagrams and obtain special classes of Venn diagrams [4]. The exact definitions used in this paper are given in Section 2.

Grünbaum wrote the following [3]: "Venn diagrams were introduced by J. Venn in 1880 [6] and popularized in his book [7]. Venn did consider the question of existence of Venn diagrams for an arbitrary number n of classes, and provided in [6] an inductive construction of such diagrams. However, in his better known book [7], Venn did not mention the construction of diagrams with many classes; this was often mistakenly interpreted as meaning that Venn could not find such diagrams, and over the past century many papers were published in which the existence of Venn diagrams for n classes is proved." Venn's inductive construction is an extension of an existing Venn diagram by the addition of a suitable simple closed curve. This method works as long as the Venn diagram is a reducible Venn diagram (see Section 2).

In 1984 Grünbaum proved the following theorem [3]: For every n $ 5, there exist simple and irreducible Venn diagrams for n classes. In 1984 Peter Winkler conjectured the following [9]: Every simple Venn diagram can be extended to a new simple Venn diagram by the addition of a suitable curve. He also showed that every reducible, simple Venn diagram can be extended to a simple new Venn diagram by the addition of a suitable simple closed curve. Finally, in 1992 Grünbaum [4] modified Peter Winkler's conjecture by dropping the word simple, and conjectured that every Venn diagram of n curves can be extended to a Venn diagram of n+1 curves by the addition of a suitable simple closed curve. In this paper we prove Grünbaum's conjecture. Winkler's conjecture is still open.

2. *Definitions and Notation*: We use graph theory, and we define some of the terms that are needed. The reader is referred to [1] or any other standard book on graph theory.

A graph that can be drawn in the plane so that its edges are Jordan arcs whose endpoints are vertices and no two edges meet except at vertices is called a *planar graph*. Such a drawing is a *plane graph* G, which can be viewed as a subset of $\mathbf{R}^2$, and $\mathbf{R}^2$-G is a union of regions called *faces* of G.

A cycle of the graph G bounding a face is called a *facial cycle*. A cycle of the graph G is a *separating cycle* if its removal (the vertices and the edges) results in a disconnected graph.

A *simple triangulation* is a connected planar graph all of whose facial walks have length three.

We follow Grünbaum [4] in the terminology of Venn diagrams. A *Venn diagram* in the plane is a collection of simple closed Jordan curves ö = $\{C_1, C_2, ..., C_n\}$ such that each of the 2^n sets X_1 1 X_2 1...1 X_n is a nonempty and connected region; here, X_i is either the bounded interior or the unbounded exterior of C_i, $i = 1,2,...,n$. We note that each of the 2^n sets can be described by an n-tuple of zeros and ones where the i^{th} coordinate is a zero if X_i is the unbounded exterior of C_i, otherwise it is one, $i = 1,2,...,n$. It is clear that there is a one-one correspondence between the 2^n sets of a Venn diagram and the vertices of the n-dimensional hypercube. If A = X_1 1 X_2 1 ... 1 X_n is a set in a Venn diagram then the corresponding n-tuple in the hypercube is the *description* of A.

A Venn diagram is a *simple Venn diagram* if at most two curves intersect (transversally) at any point in the plane. Among the nonsimple Venn diagrams, we shall consider only those in which any two curves meet (not necessarily transversally) in points and not in segments of curves. With this restriction on the Venn diagrams it is easy to see that if two sets A and B in a Venn diagram share a common boundary (i.e., a segment of a curve) then their descriptions must differ exactly in one coordinate. Also it is not hard to see that if two sets A and B differ in their descriptions in two or more coordinates then the sets cannot share a segment of a curve as their common boundary.

The projection of a Venn diagram from the plane to the sphere via stereographic projection yields a spherical Venn diagram. In this article there is no loss of generality in working with spherical Venn diagrams; accordingly we assume all the Venn diagrams are drawn on a sphere. By doing this all of the faces of the Venn diagrams will be finite and simply connected.

A Venn diagram ö with n curves is called irreducible if each of the n families of n-1 curves, obtained from ö by deleting in turn one of the n curves, fails to be a Venn diagram. Otherwise it is called reducible.

We can associate three distinct graphs (or multigraphs) with any Venn diagram on a set of n curves ö = $\{C_1,C_2,...,C_n\}$. The Venn diagram itself can be viewed as a graph V(ö) where all the intersection points of the curves in ö are the vertices and the segments of the curves with vertices as the end points are the edges. We refer to this graph simply as the Venn diagram. The Venn diagram V(ö) is a planar graph but it may have multiple edges.

The second graph is the dual graph of V(ö), denoted by D(ö). We note that D(ö) depends on how the Venn diagram is drawn. We will call this the Venn graph. We later show that the Venn graph D(ö) is a simple planar graph. Note that two vertices of the Venn graph are adjacent only if their descriptions differ exactly at one coordinate. Consequently if ö is a Venn diagram then the dual graph D(ö) is a subgraph of the hypercube and hence D(ö) is a bipartite graph.

The third graph we introduce is the radual graph D^*(ö), which is the union of the dual graph and the radial graph, see [5]. This graph is obtained by adding a new vertex x_F in each face F of the dual graph D(ö) and joining each x_F to each vertex y of F by one edge for each appearance of y in the facial walk of F. We will call the new vertices radial vertices, and the vertices of the Venn graph dual vertices. Note that the radial vertex set can be viewed as the vertex set of the Venn diagram. Note also that there are no edges between radial vertices.

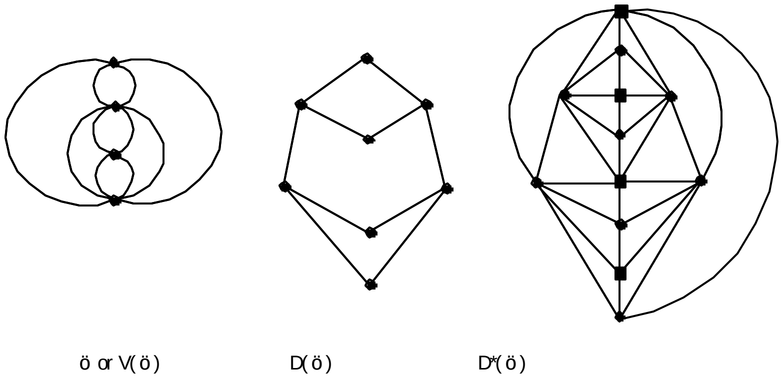
ö or V(ö) D(ö) D*(ö)

Figure 1

We prove first that the radual graph associated with a Venn diagram is Hamiltonian. We then show that the Hamilton cycle itself as a new curve extends the Venn diagram to a new Venn diagram, thus proving the main result of this paper. An example of a Venn diagram with three curves and the corresponding Venn graph and radual graph are shown in Figure 1.

3. Properties of Venn graphs and Radual Graphs: We now prove that the radual graph of a Venn diagram is a simple triangulation with no separating triangles. To achieve this goal we need several properties of the dual and radual graphs of a Venn diagram. Some of these properties are simple observations, however Lemma 1 is a fundamental property of Venn diagrams and is used in the enumeration of Venn diagrams on three curves in [2]. Here, we need the same result to prove several properties of dual and radual graphs and we include a modified short version of the proof of Lemma 1 for the completeness of this paper. Throughout this section we assume *ö* $ 3.

Lemma 1: No two edges in a face in a Venn diagram belong to the same curve.

Proof: Suppose there is a face f in a Venn diagram V(ö) having four or more edges, at least two of which, say e_1 and e_2, belong to the same curve C_i in ö. First suppose that the edges e_1 and e_2 are not adjacent. Let p and q be two points outside the face f but in a neighborhood of

e_1 and e_2 respectively. The regions containing p and q have the same description, differing from the description of f exactly in the i^{th} coordinate, and hence p and q are contained in a single face h of V(ö). There is a path ' contained in h joining p and q, which does not intersect any edges of V(ö). Let $'_1$ be a path joining p and q passing through the face f. The closed curve
$' c '_1$ intersects the curve C_i and does not intersect any other curve in ö. Since e_1 and e_2 are not adjacent, it is easy to see that there must exist two distinct curves C_j and C_k (different from C_i) in V(ö), such that C_j is in the interior of $'_1 c '$ and C_k is in the exterior, contradicting the fact that V(ö) is a Venn diagram. Now suppose the edges e_1 and e_2 are adjacent at a vertex x of V(ö). Then choosing the points p and q arbitrarily close to x, it is easy to see that one of the interior or the exterior of C_i does not intersect at least one curve that passes through x, contradicting the fact that V(ö) is a Venn diagram. In this last case we have not used the fact that the face f has four or more edges, and so it is easy to see that Lemma 1 is also true if a face has fewer than four edges. $

Theorem 2: A Venn graph has no multiple edges.

Proof: Suppose a Venn graph D(ö) has multiple edges between the vertices x and y. Let X and Y be faces of V(ö) corresponding to x and y respectively. Since x and y are adjacent, the descriptions of the faces X and Y must differ in exactly one coordinate, say the j^{th}

coordinate. Thus one of the regions X and Y is in the interior of C_j and the other is in the exterior of C_j. This implies that all the disjoint segments of curves forming the common boundary of X and Y belong to the same curve C_j. Since x and y have multiple edges, the faces X and Y each have at least two segments of the same curve C_j on the common boundary between them. This contradicts Lemma 1. $

Theorem 3: The deletion of any two adjacent vertices of a Venn graph does not disconnect the graph.

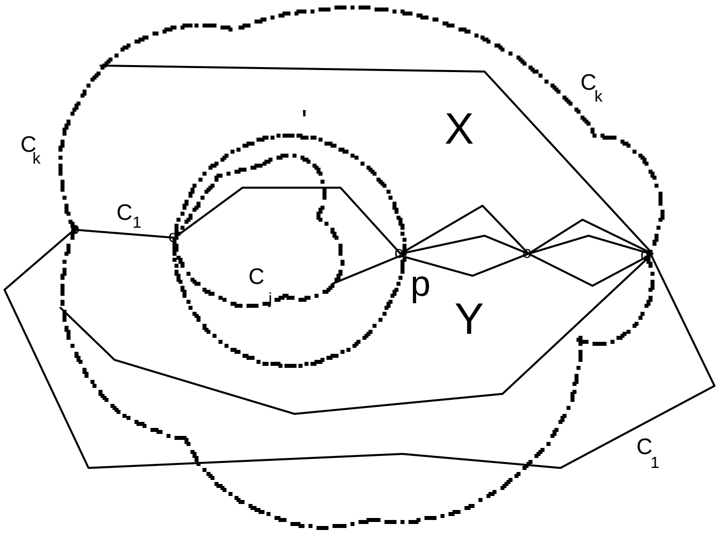

Figure 2

Proof: Suppose x and y are two adjacent vertices in a Venn graph D(ö) whose removal disconnects D(ö). Let X and Y be the faces of V(ö) corresponding to the vertices x and y respectively in D(ö). Then it follows from Lemma 1 and Theorem 2 that X and Y have one and only one edge (one segment of a curve in ö) in common. Also the removal of faces

X and Y together with their boundaries from the plane separates the plane into two or more regions. Let us assume that the common edge belongs to curve C_1. Then the boundary of the region X c Y has a segment of the curve C_1 and at least one point (or possibly more, but a finite number of points) in common as shown in Figure 2. The curve C_1 must pass through these points. There exists a simple Jordan curve ' contained entirely in the regions X and Y and their common boundaries as shown in Figure 2. We can choose the curve ' so that it intersects the common boundary segment of C_1 and a common boundary point p. There are curves C_j and C_k in ö that intersect the curve C_1 at the two ends of the common edge of the face X and Y. The segments of the curves C_j and C_k shown (only schematically) in Figure 2 must be distinct from C_1 (Lemma 1), and from each other, for if not, the point p is a self intersection point of a curve in ö which is impossible. Since the only exit point of C_j from the interior of ' is p, C_j is either entirely inside of C_k or entirely outside of C_k. This contradicts the fact that V(ö) is a Venn diagram. $

We note that in the hypothesis of Theorem 3 the adjacency of the vertices x and y is important, as can be seen by the Venn diagram on two curves for which the 4-cycle is the Venn graph which can be disconnected by deleting any two nonadjacent vertices.

Theorem 4: A Venn graph is simple and nonseparable.

Proof: By Theorem 2, the Venn graph D(ö) has no multiple edges, and it obviously has no loops. It follows from Theorem 3 that it has no cut vertices. $

Corollary 5: All facial walks in a Venn graph are simple closed walks.

Proof: Suppose there is a face f in D(ö) whose facial walk is not simple. Let x be a vertex that repeats in the facial walk of the face f. Then x is a cut vertex of D(ö), contradicting Theorem 4. $

Theorem 6: The radual graph of a Venn diagram is a simple triangulation with no separating triangles.

Proof: Since the Venn graph D(ö) is simple and all facial walks of D(ö) are simple, it follows that the radual graph D^*(ö) is a simple graph. From the construction of the radual graph D^*(ö) it follows immediately that it is a simple triangulation. Suppose that x, y, and z are any three vertices of the radual graph D^*(ö) forming a triangle in D^*(ö). We will show that the triangle xyz is not a separating triangle of D^*(ö). Since D(ö) is a bipartite graph , not all of the vertices {x,y,z} can belong to D(ö). Furthermore no two radial vertices are adjacent in the radual graph. Thus exactly one of the three vertices is a radial vertex and the other two are dual vertices.

Without loss of generality let us assume that the vertices x and y are dual vertices and z is a radial vertex. Let u and v be two vertices not in {x,y,z}. If both u and v are dual vertices, then by Theorem 3 there is a path joining u and v which avoids the set {x,y,z}, since the graph D(ö) cannot be disconnected by deleting the adjacent vertices x and y. If either u or v or both are radial vertices, then again we can find a path joining u and v that avoids the set {x,y,z} since each radial vertex is adjacent to at least three dual vertices because D(ö) has no two-faces. Thus D^*(ö) has no separating triangles. $

4. <u>Proof of the Main Result</u>: We need the following classical theorem of Whitney concerning Hamilton cycles in simple triangulations.

<u>Theorem</u> (Whitney [8]): A simple triangulation without separating triangles is Hamiltonian.

<u>Theorem 7</u>: The radual graph of a Venn diagram is Hamiltonian.
<u>Proof</u>: If *ö* = 1 then the radual graph D^*(ö) is a triangle. If *ö* = 2, then D(ö) is a 4-cycle and is unique. The Hamiltonicity of D^*(ö) is easy to see in this case. If *ö* $ 3, then the radual graph D^*(ö) is a simple triangulation, without separating triangles by Theorem 6, so Whitney's Theorem implies that it has a Hamilton cycle. $

Theorem 8: Every Venn diagram with n curves can be extended to a new Venn diagram with n+1 curves by the addition of a suitable curve.

Proof: Let ö be a Venn diagram with n curves. By Theorem 7, the radual graph D^*(ö) is Hamiltonian. The Hamilton cycle itself is the new curve extending the Venn diagram ö. Since the Hamilton cycle passes through every vertex of D(ö) exactly once, thus dividing every region of the Venn diagram into exactly two connected regions, the resulting family of n+1 curves is a Venn diagram. $

The approach of this paper necessarily results in nonsimple Venn diagrams and therefore this approach is not likely to be useful for Winkler's conjecture.

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