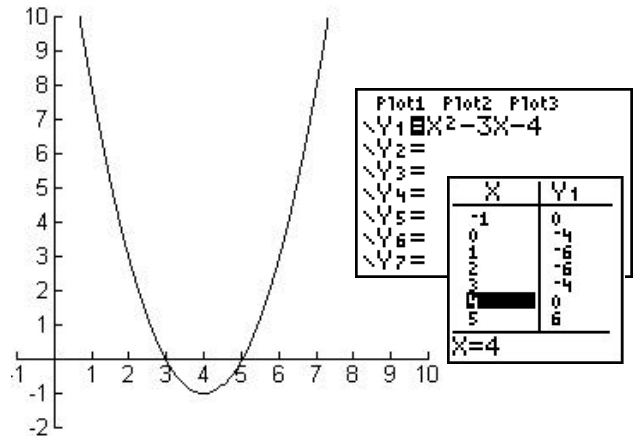


Linear Factors and Zeros of Polynomials

If $f(x) = x^2 - 8x + 15 = (x-3)(x-5)$ then $x-3$ and $x-5$ are factors of $f(x)$.

If we set $f(x)$ equal to zero, then we have:

$$\begin{aligned} f(x) &= 0 \\ x^2 - 8x + 15 &= 0 \\ (x-3)(x-5) &= 0 \\ x-3 = 0 \text{ or } x-5 &= 0 \\ x = 3 \text{ or } x = 5 \end{aligned}$$



We say that $x = 3$ and $x = 5$ are zeros of $f(x)$.

If we graph $f(x)$, it has x -intercepts at $(3, 0)$ and $(5, 0)$.

The following statements are equivalent:

1. $x = a$ is a zero of the function $f(x)$.
2. $x = a$ is a solution (root) to the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a factor of $f(x)$.
4. $(a, 0)$ is an x -intercept of the graph of $f(x)$.

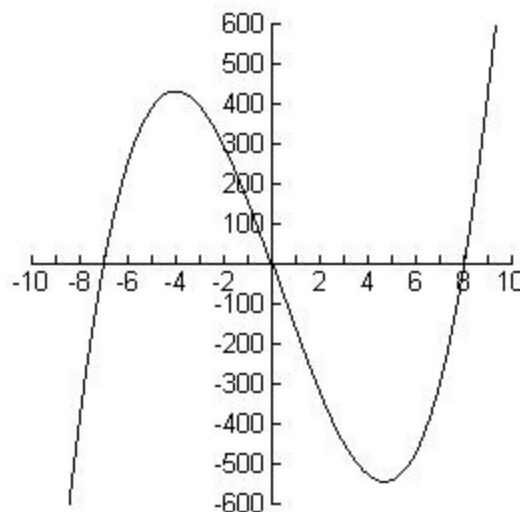
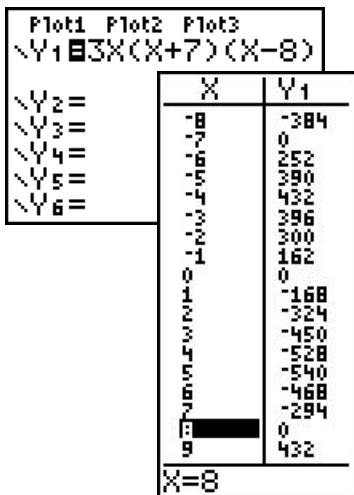
Example 1.

Given $f(x) = 3x(x+7)(x-8)$

The factors of $f(x)$ are: $3x$, $x+7$, $x-8$

The zeros of $f(x)$ are: $x=0$, $x=-7$, $x=8$

The graph of $f(x) = 3x(x+7)(x-8)$ crosses the x -axis at $(0,0)$, $(-7,0)$, and $(8,0)$.



Example 2.

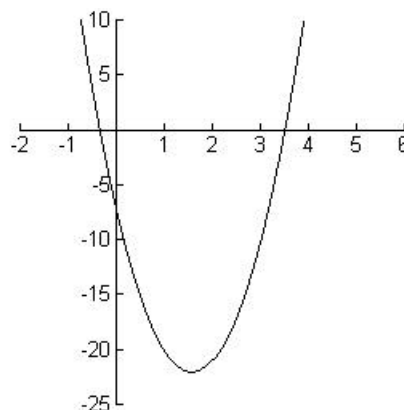
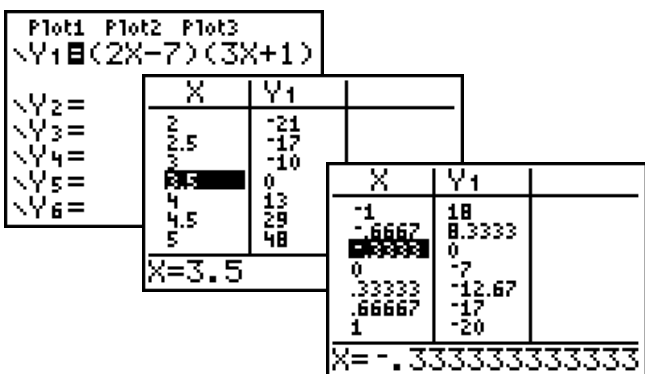
Given $f(x) = (2x - 7)(3x + 1)$

The factors of $f(x)$ are: $2x - 7$ and $3x + 1$

The zeros of $f(x)$ are: $2x - 7 = 0 \Rightarrow x = \frac{7}{2}$

$$3x + 1 = 0 \Rightarrow x = -\frac{1}{3}$$

The graph of $f(x)$ crosses the x -axis at $(3.5, 0)$ and $(-0.33\bar{3}, 0)$.



Example 3.

Find the zeros of $f(x) = x^3 + 4x^2 - 12x$

Solution:

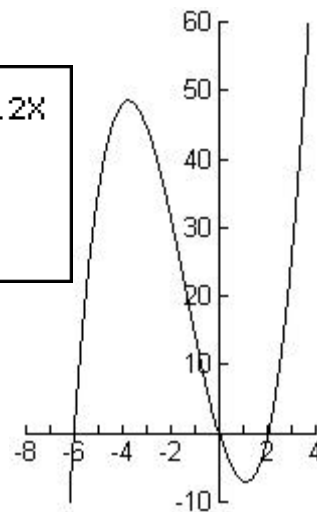
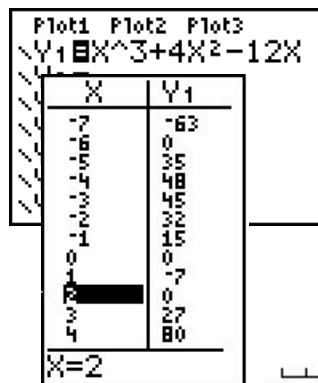
$$x^3 + 4x^2 - 12x = 0$$

$$x(x^2 + 4x - 12) = 0$$

$$x(x+6)(x-2) = 0$$

$x = 0$ or $x + 6 = 0$ or $x - 2 = 0$
 $x = 0$ or $x = -6$ or $x = 2$

The zeros of $f(x)$ are: $x = 0, -6,$ and 2 .



Example 4.

Find a polynomial with zeros $x = 4$ and -1 .

$x = 4 \Rightarrow x - 4$ is a factor.

$x = -1 \Rightarrow x + 1$ is a factor.

$$f(x) = (x - 4)(x + 1) = x^2 - 3x - 4$$

Note: The answer is not unique. Any non-zero multiple of $f(x)$ also has these zeros, such as

$$g(x) = 5x^2 - 15x - 20.$$

Example 5

Find a polynomial with zeros $x=0$, $\frac{3}{4}$, and $-\frac{2}{9}$.

Solution: $x=0 \Rightarrow x-0$ is a factor, or, more simply, x is a factor.

$x=\frac{3}{4} \Rightarrow 4x-3$ is a factor.

$x=-\frac{2}{9} \Rightarrow 9x+2$ is a factor.

$$\begin{aligned} f(x) &= x(4x-3)(9x+2) \\ &= x(36x^2-19x-6) \\ &= 36x^3-19x^2-6x \end{aligned}$$

Again, this answer is not unique.

$g(x) = x\left(x-\frac{3}{4}\right)\left(x+\frac{2}{9}\right)$ also has these zeros, but does not have nice integer coefficients.

Example 6.

Find all the x -intercepts of $f(x) = 4x^3 - 100x$

$$= 4x(x^2 - 25)$$

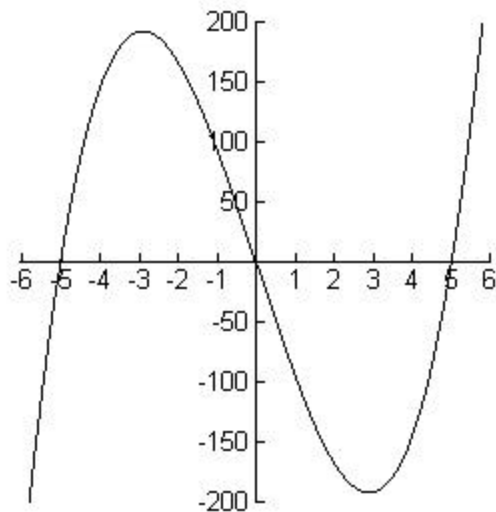
$$= 4x(x+5)(x-5)$$

The zeros of $f(x)$ are $x = 0$, -5 , and 5 . The x -intercepts of the graph of $f(x)$ are $(0, 0)$, $(-5, 0)$, and $(5, 0)$.

Plot1	Plot2	Plot3
Y1	4X^3-100X	
Y2	=	
Y3	=	
Y4	=	
Y5	=	
Y6	=	
Y7	=	

X	Y1
-6	-264
-5	0
-4	144
-3	192
-2	168
-1	96
0	0
1	-96
2	-168
3	-192
4	-144
5	0
6	264

X=5



Example 7

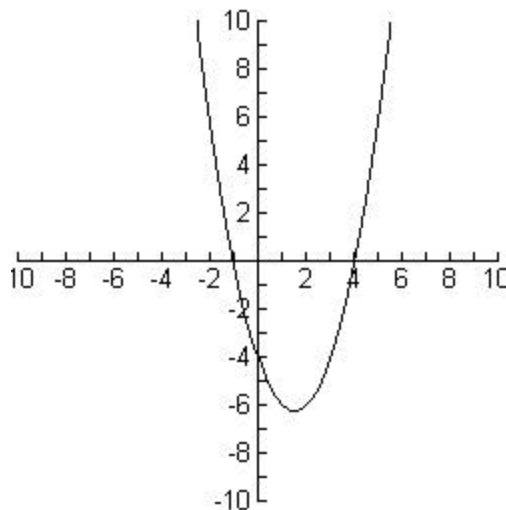
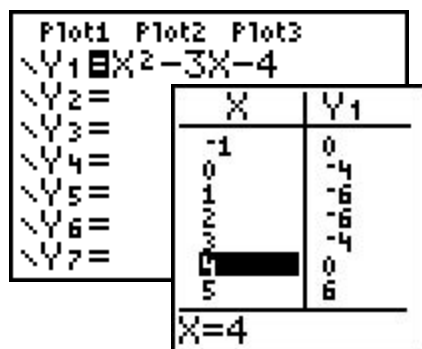
Find a function that has x -intercepts at $(4, 0)$ and $(-1, 0)$. (Answer is not unique.)

Solutions: The zeros of $f(x)$ are $x = 4$ and $x = -1$.

The factors of $f(x)$ are $(x-4)$ and $(x+1)$.

$$\begin{aligned} \text{Let } f(x) &= (x-4)(x+1) \\ &= x^2 - 3x - 4 \end{aligned}$$

Check: The graph of $f(x) = x^2 - 3x - 4$ crosses the x -axis at $(4,0)$ and $(-1,0)$.

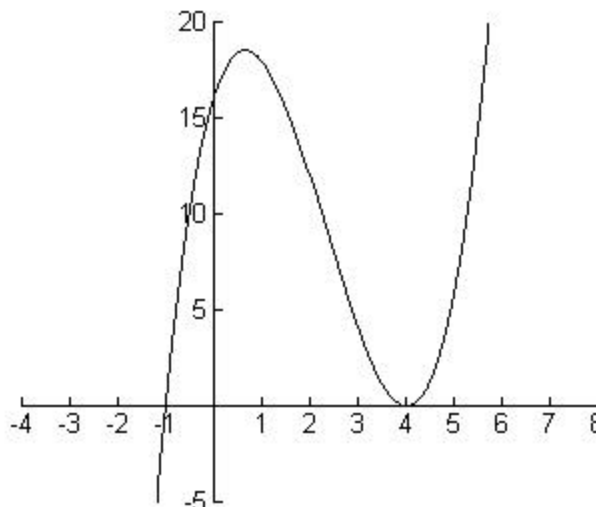
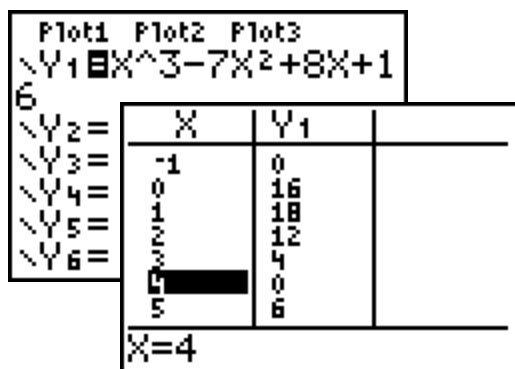


Note: $f(x) = x^2 - 3x - 4$ also has x -intercepts at $(4, 0)$ and $(-1, 0)$.

Example 8

Use the graph of $f(x) = x^3 - 7x^2 + 8x + 16$ to find the zeros of the function.

Solution:



From the graph we see that $f(x)$ has x -intercepts at $(4, 0)$ and $(-1, 0)$. The zeros of $f(x)$ and $x = 4$ and -1 . **Note:** This function is also another solution to Example 7.

Exercises

1. Find all the real zeros of the function, and then list all the x -intercepts of the function.

a) $f(x) = 7x^2(x+4)(x-9)$

b) $g(x) = 3(x+2)(x-8)(x+5)$

c) $h(x) = x^2 + 4$

d) $f(x) = x^2 - 4$

e) $g(x) = 9x^2 - 64$

f) $h(x) = x^2 - 8x - 9$

g) $f(x) = x^2 + 4x - 21$

h) $f(x) = x^3 - 3x^2 - 10x$

i) $g(x) = 3x^2 - 2x - 8$

j) $g(x) = 6x^2 + 5x + 1$

k) $h(x) = 4x^3 + 2x^2 - 6x$

l) $h(x) = 4x^3 - 36x$

2. Find a polynomial with integer coefficients with the given zeros and then use a graphing calculator to check your answer.

a) $x = 0, x = 3$

b) $x = 1, x = -4$

c) $x = 0, x = -6, x = 6$

d) $x = \frac{2}{3}, x = 4$

e) $x = \frac{3}{4}, x = -\frac{2}{5}$

Note: The Department of Mathematical Sciences Web Site (<http://www.ipfw.edu/math/>) maintains an **Assistance with Graphing Calculators** Web Page at <http://www.ipfw.edu/math/graphcalc.html> to help you obtain and use a graphing calculator. In particular, there is a link to a page of Graphing Calculator Instructions providing step-by-step help for common commands (window set-up, finding x -intercepts, using a table, etc.) for many models of graphing calculators.