A VARIANT OF HOPF LEMMA FOR SECOND ORDER DIFFERENTIAL INEQUALITIES

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Abstract. We prove a sequence version of Hopf lemma, which is essentially equivalent to the classical version. We show the applications of the new version, especially in characterizing zeros of solutions of second order ordinary differential inequalities.

The classical Hopf Lemma is a fundamental result in the theory of differential equations. In this paper, we prove a Hopf Lemma with weaker assumptions and we will show its applications, especially its usefulness in characterizing zeros of solutions of second order ordinary differential inequalities. Our work is motivated in part by recent research and development in this area [2, 3, 4].

In the first section we state main results and two examples. The proofs of the main theorem and its lemmas are provided in the next section, followed by a section containing the proofs of the rest results.

1. Classical Hopf Lemma and main results

The classical Hopf Lemma can be stated as the following [5].

Hopf Lemma. Let $u \in \mathcal{C}^2(a,b)$, $u \in \mathcal{C}^1(a,b]$ *and* $u''(x)+\alpha(x)u'(x)+\beta(x) \le 0$ *for* $x \in (a,b)$*, where* $|\alpha(x)| \le C$, $|\beta(x)| \le C$ *for some constant* C*. If* $u(x) > 0$ *for* $x \in (a,b)$ *and* $u(b)=0$*, then* $u'(b) < 0$.

The main result Theorem 1 is a sequence version of the Hopf Lemma. First we state a concise version parallel to the stated classical version.

Theorem 1*. *Let* $u \in \mathcal{C}^2(a,b)$, $u \in \mathcal{C}^1(a,b]$*. Assume*

$$L(u)(x) = u''(x)+\alpha(x)u'(x)+\beta(x)u(x) \le 0, \qquad x \in (a,b)$$

where $|\alpha(x)|$, $|\beta(x)| \le C$*. If there exists a sequence*

$$\{x_k\} \subset (a,b), \quad x_k \to b, \quad u(x_k) > 0 \qquad \text{and} \quad u(b)=0,$$

then $u'(b) < 0$.

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The more general version with weaker assumptions is stated below as Theorem 1, which is a more practical form for applications. We also added the statement corresponding to the left endpoint of the interval. Theorem 1^* is a special case of Theorem 1 when u' exists at the endpoint.

Theorem 1. *Let* $u \in \mathcal{C}^2(a,b)$. *Assume*

$$L(u)(x) = u''(x) + \alpha(x)u'(x) + \beta(x)u(x) \leq 0, \qquad x \in (a,b) \tag{1}$$

where $|\alpha(x)|,\ |\beta(x)| \leq C$. *If* $u \in \mathcal{C}(a,b]$ *and if there exists a sequence*

$$\{x_k\} \subset (a,b), \quad x_k \to b, \quad u(x_k) > 0 \qquad \textit{and} \quad u(b) = 0,$$

then $\displaystyle\limsup_{x\to b-} \frac{u(x)}{x-b} < 0$. *If* $u \in \mathcal{C}[a,b)$ *and if there exists a sequence*

$$\{x_k\} \subset (a,b), \quad x_k \to a, \quad u(x_k) > 0 \qquad \textit{and} \quad u(a) = 0,$$

then $\displaystyle\limsup_{x\to a+} \frac{u(x)}{x-a} > 0$.

It is clear that if the classical Hopf lemma holds with $u(x) > 0$ on (a,b), then Theorem 1 and Theorem 1^* hold. Corollary 2 shows that the reverse is true in a neighborhood of b, therefore Theorem 1 (consequently Theorem 1^*) and the classical Hopf lemma are essentially equivalent.

Corollary 2. *Let* $u \in \mathcal{C}^2(a,b),\ u \in \mathcal{C}(a,b]$. *Assume* $u(b) = 0$ *and (1) holds. If there is a sequence* $\{x_k\} \subset (a,b)$ *such that* $x_k \to b,\ u(x_k) > 0$, *then there exists* $\delta > 0$ *such that* $u(x) > 0$ *for* $x \in (b-\delta, b) \subset (a,b)$.

Theorem 1 is useful in studying zeros of solutions of differential inequalities. First we define transverse zeros.

Definition 3. Let $f(x)$ be a continuous function on $[a,b]$, $f \in \mathcal{C}^1(a,b)$. If $x_0 \in (a,b), f(x_0) = 0, f'(x_0) \neq 0$, then x_0 is called a transverse zero.

We relax the definition of transversality at an endpoint where the one-sided derivative may not exist. If $f(b) = 0$, if there exists $\delta > 0$ such that $f(x) > 0$, $\forall x \in (b-\delta, b)$ and $\limsup_{x\to b-} \frac{f(x)-f(b)}{x-b} < 0$, or $f(x) < 0$, $\forall x \in (b-\delta, b)$ and $\limsup_{x\to b-} \frac{f(x)-f(b)}{x-b} > 0$, then b is a transverse zero of f. The transversality at $x_0 = a$ is defined similarly.

Denote

$$J_0(f) = \{x \in [a,b], f(x) = 0\}$$
$$J_+(f) = \{x \in [a,b], f(x) > 0\}$$
$$J_-(f) = \{x \in [a,b], f(x) < 0\}$$

From the definitions it is clear that J_+, J_- are unions of disjoint open intervals (half open at boundaries). Since at a transverse zero in (a, b) the function must cross x axis from $f < 0$ to $f > 0$ or from $f > 0$ to $f < 0$, a transverse zero in (a, b) must be a common endpoint of an interval in J_+ and an interval in J_-.

Hopf Lemma gives the condition when a solution has a transverse zero (not a cross zero with $f' = 0$). The following theorem characterizes the behavior of zeros of the solutions of differential inequality (1), especially the uniform finiteness of the number of transverse zeros. The conclusion of Theorem 4 shows that solutions of differential inequality (1) possess no isolated non-transverse zeros, thus clarifies certain issues in previous results on zeros of solutions of differential equations as discussed in [1] (and there mentioned [6, 7]).

Theorem 4. *Let $u \in \mathcal{C}^2(a, b)$, $u \in \mathcal{C}[a, b]$ and (1) holds. Let N_-, N_+, N_0 denote the numbers of intervals in $J_-(u)$, $J_+(u)$, $J_0(u)$ respectively, and let Z_0 be the number of transverse zeros of u in $[a, b]$. Then N_-, N_+, N_0 and Z_0 are finite, and*

$$N_- \le (b-a)L_C, \qquad |N_+ - N_-| \le 1, \qquad |N_0 - N_-| \le 1, \qquad |Z_0 - N_+ - N_-| \le 1,$$

where $L_C < \infty$ is a constant depending only on C, and $L_C \le \frac{(4C+2)}{\ln(2C+2)}$.

Theorem 4 can be in term of non-homogeneous, nonlinear ordinary differential equations as stated in Theorem 5 below. Although finiteness of zeros of solutions of homogeneous ordinary differential equations can be obtained from the uniqueness theorem, it is nontrivial to obtain a uniform bound for the number of transverse zeros.

Theorem 5. *Let $u \in \mathcal{C}^2(a, b)$, $u \in \mathcal{C}[a, b]$ satisfy the differential equation*

$$u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = f(x, u), \qquad x \in (a, b)$$

with $|\alpha(x)|, |\beta(x)| \le C$. If

$$f(x, y) \le 0, \qquad \forall (x, y) \in (a, b) \times \mathbb{R},$$

then for any solution u, the number of transverse zeros of u in $[a, b]$ is bounded by $2(b-a)L_C + 2$, where L_C is a constant depending only on C, and $L_C \le \frac{(4C+2)}{\ln(2C+2)}$.

Corollary 6 below describes the convexity of u in Theorem 1 at certain critical points.

Corollary 6. *Assume (1) holds for $u \in \mathcal{C}^2(a, b)$, $u \in \mathcal{C}[a, b]$. If $u(x_0) = u'(x_0) = 0$ for some $x_0 \in (a, b)$, then $u(x_0)$ is a local maximum.*

Notice that under the assumptions of Corollary 6, it is immediate that $L(u)(x_0) = u''(x_0) \le 0$ by $L(u) \le 0$. Applying Theorem 1 excludes the case of x_0 being an inflection point to obtain the maximum property in Corollary 6.

More general critical points are discussed in Corollary 7 below, which shows that $L(u) \leq 0$ is not the same as convexity.

Corollary 7. *Assume (1) holds for* $u \in \mathcal{C}^2(a,b),\ u \in \mathcal{C}[a,b]$. *If* $u'(x_0) = 0$ *for some* $x_0 \in (a,b)$ *and* $b(x)u(x_0) \geq 0$ *in a neighborhood of* x_0, *then* $u(x_0)$ *is a local maximum.*

Theorem 1, Theorem 4 and Corollary 6 can be applied to nonlinear functionals as stated in Theorem 8 and Corollary 9 below.

Theorem 8. *Let*

$$F(x_1, x_2, x_3, x_4) \in \mathcal{C}^1(\mathbb{R}^4), \qquad \frac{\partial F}{\partial x_4} > 0.$$

If $u, v : [a,b] \to \mathbb{R}, u, v \in \mathcal{C}^2(a,b),\ u, v \in \mathcal{C}[a,b]$ *satisfy*

$$F(x, u, u', u'') \leq F(x, v, v', v''), \qquad x \in (a,b)$$

and $u(x_0) = v(x_0),\ u'(x_0) = v'(x_0)$ *for* $x_0 \in (a,b)$, *then* $u(x) \leq v(x)$ *in a neighborhood of* x_0.

A point x_0 is a transverse intersection of two functions u and v if and only if x_0 is a transverse zero of $w = u - v$.

Corollary 9. *Let*

$$F(x_1, x_2, x_3, x_4) \in \mathcal{C}^1(\mathbb{R}^4), \qquad \frac{\partial F}{\partial x_4} > 0.$$

If $u, v : [a,b] \to \mathbb{R}, u, v \in \mathcal{C}^2(a,b),\ u, v \in \mathcal{C}[a,b]$ *satisfy*

$$F(x, u, u', u'') \leq F(x, v, v', v''), \qquad x \in (a,b),$$

then the number of transverse intersections of u *and* v *is bounded by* $2(b-a)L_C + 2$, *and the number of* $u \equiv v$ *intervals is bounded by* $(b-a)L_C + 1$, *where* L_C *is a constant depending only on* C, *and* $L_C \leq \frac{(4C+2)}{\ln(2C+2)}$.

The following two examples show the type of applications of the above results.

Example 1. For $n > 0$, the function

$$u(0) = 0, \qquad u(x) = x^n \sin\frac{1}{x}$$

is a solution of the differential equation

$$L(u) = u'' + \alpha(x)u' + \beta(x)u = 0$$

with

$$\alpha(x) = \frac{2-2n}{x}, \qquad \beta(x) = \frac{1 + (n^2 - n)x^2}{x^4}.$$

In addition, $u \in \mathcal{C}^2(a,0),\ u \in \mathcal{C}[a,0]$ for any $a < 0$. For any $b \in (a,0)$, the absolute values of the coefficients $|\alpha(x)|$ and $|\beta(x)|$ are bounded by C on interval (a,b) for any $C \geq (1+(n^2-n)b^2)/b^4$. The conclusions of Theorem 1 and Theorem 4 and the corollaries hold.

Since $\alpha(x)$ and $\beta(x)$ are not bounded near $x=0$, Theorem 1 and Theorem 4 do not apply to u on the interval $(a,0)$. Notice that $u'(0)=0$, and there are infinitely many transverse zeros in $(a,0)$.

Example 2. The differential inequality

$$L(u) = u'' \leq 0, \qquad x \in (-1,1)$$

has many solutions. One of them is

$$u(x) = \begin{cases} 0, & x \leq 0, \\ -x^p, & x > 0, \end{cases} \qquad p \in (2,\infty),$$

which satisfies the non-homogeneous, non-Lipschitz differential equation

$$u'' = -p(p-1)|u|^{1-2/p} \leq 0, \qquad u \in \mathcal{C}^2(-1,1),\ u \in \mathcal{C}[-1,1].$$

u satisfies the assumptions in Theorem 1, Theorem 4 and the corollaries for any $C \geq 0$, therefore the conclusions hold. Notice that u has no transverse zeros and there are uncountably many non-transverse zeros. However the non-transverse zeros form a closed interval $[-1,0] \subset [-1,1]$.

2. Proofs of lemmas and Theorem 1

We need three lemmas to assist the proof of the main result Theorem 1.

First we explicitly construct a function h with $h'(b) > 0$. Eventually we will prove Theorem 1 by showing $u'(b) \leq -\epsilon h'(b) < 0$.

Lemma 10. *Let*

$$L(u)(x) = u''(x) + \alpha(x)u'(x) + \beta(x)u(x), \quad |\alpha(x)|,\ \beta(x)| \leq C, \quad x \in (a,b).$$

Consider the function

$$h(x) = e^{-\gamma(x-b)} - 1, \qquad x \in [a,b].$$

If $\gamma \geq C+1$, *then*

$$h(b) = 0 \qquad \textit{and} \qquad h(x) > 0,\ L(h) > 0 \quad \textit{for } x \in (a,b).$$

Proof of Lemma 10. Calculation shows that $h(b) = 0$ and $h(x) > 0$ for $x \in (a, b)$.

$$\begin{aligned} L(h)(x) &= h''(x) + \alpha(x)h'(x) + \beta(x)h(x) \\ &= \gamma^2 e^{-\gamma(x-b)} - \alpha(x)\gamma e^{-\gamma(x-b)} + \beta(x)\left(e^{-\gamma(x-b)-1}\right) \\ &\geq e^{-\gamma(x-b)}\left[\gamma^2 - |\alpha(x)|\gamma - |\beta(x)|\left(1 - e^{\gamma(x-b)}\right)\right] \\ &\geq e^{-\gamma(x-b)}\left(\gamma^2 - C\gamma - C\right) \end{aligned}$$

using $|1 - e^{\gamma(x-b)}| < 1$ and $|\alpha(x)|, |\beta(x)| \leq C$ in the last inequality. When $C = 0$, it is clear that $L(h) > 0$ for any $\gamma > 0$. For $C > 0$, by the quadratic formula,

$$\gamma^2 - C\gamma - C > 0 \qquad \textit{if} \qquad \gamma > \frac{C + \sqrt{C^2 + 4C}}{2} = \frac{C}{2} + \frac{C}{2}\sqrt{1 + 4/C}$$

By Taylor's formula for $\sqrt{1+x}$ at 0,

$$\sqrt{1 + 4/C} = 1 + 2/C - 2(1+\xi)^{-3/2}/C^2 < 1 + 2/C, \quad \textit{where} \quad \xi \in (0, 4/C).$$

Hence

$$\frac{C}{2} + \frac{C}{2}\sqrt{1 + 4/C} < C + 1.$$

Therefore for the given C,

$$\gamma \geq C + 1 \quad \Longrightarrow \quad \gamma^2 - C\gamma - C > 0 \quad \Longrightarrow \quad L(h) > 0$$

This concludes the proof of Lemma 10. □

Next, we construct an auxiliary function m such that $m(x) > 0$ on an interval $(b - \eta, b] \subset (a, b]$.

Lemma 11. *Let*

$$L(u)(x) = u''(x) + \alpha(x)u'(x) + \beta(x)u(x), \qquad |\alpha(x)|,\ \beta(x)| \leq C, \qquad x \in (a, b).$$

Consider the function

$$m(x) = e^{\gamma\eta} - e^{\gamma(x-b)}, \qquad x \in [a, b].$$

If

$$\gamma = 2C + 1, \qquad \eta \leq \min\left\{\frac{\ln(2C+2)}{4C+2},\ b - a\right\}$$

then

$$m(b) > 0 \qquad \textit{and} \quad m(x) > 0, \quad L(m) < 0 \quad \textit{for} \quad x \in (b - \eta, b) \subset (a, b).$$

Proof of Lemma 11. For $x \in (b-\eta, b]$, we have $x-b \le 0 < \eta$, $e^{\gamma(x-b)} < e^{\gamma\eta}$, thus $m(x) > e^{\gamma\eta} - e^{\gamma\eta} = 0$. For $x \in (b-\eta, b)$,

$$\begin{aligned} L(m)(x) &= m''(x) + \alpha(x)m'(x) + \beta(x)m(x) \\ &= -\gamma^2 e^{\gamma(x-b)} - \alpha(x)\gamma e^{\gamma(x-b)} + \beta(x)\left(e^{\gamma\eta} - e^{\gamma(x-b)}\right) \\ &\le e^{\gamma(x-b)}\left[-\gamma^2 + |\alpha(x)|\gamma + |\beta(x)|\left(e^{\gamma\eta+\gamma(b-x)} - 1\right)\right] \\ &\le e^{\gamma(x-b)}\left[-\gamma^2 + C\gamma + C\left(e^{2\gamma\eta} - 1\right)\right] \end{aligned}$$

where we applied $b - x < \eta$ to obtain the last inequality. To make the last expression < 0, we may choose $e^{2\gamma\eta} - 1 \le \gamma$ while $\gamma > 2C$, so that

$$-\gamma^2 + C\gamma + C\left(e^{2\gamma\eta} - 1\right) \le -\gamma^2 + 2C\gamma < 0.$$

Therefore for $\gamma = 2C + 1 > 2C$,

$$\eta \le \frac{\ln(2C+2)}{4C+2} = \frac{\ln(1+\gamma)}{2\gamma} \quad \Longleftrightarrow \quad e^{2\gamma\eta} \le 1 + \gamma \quad \Longrightarrow \quad L(m) < 0$$

for $x \in (b-\eta, b) \subset (a, b)$. Hence Lemma 11 holds. $\square$

The auxiliary function m in Lemma 11 is used to establish a maximum principle on small intervals near b, as stated in Lemma 12 below.

Lemma 12. *Let $u \in \mathcal{C}^2(a, b)$, $u \in \mathcal{C}(a, b]$,*

$$L(u)(x) = u''(x) + \alpha(x)u'(x) + \beta(x)u(x), \qquad |\alpha(x)|,\ \beta(x)| \le C, \qquad x \in (a, b)$$

and let

$$(d, b) \subset (a, b), \qquad a < d < b - \eta, \qquad \eta < \frac{\ln(2C+2)}{4C+2}.$$

If

$$u(d) \ge 0,\ u(b) \ge 0 \qquad \text{and} \quad L(u) \le 0 \quad \text{on} \quad (d, b),$$

then

$$u(x) \ge 0, \qquad \forall x \in (d, b).$$

Proof of Lemma 12. Let

$$z(x) = \frac{u(x)}{m(x)}, \qquad m(x) = e^{\gamma\eta} - e^{\gamma(x-b)}, \qquad x \in [d, b].$$

By Lemma 11, $m(x) > 0$ for $x \in [d, b]$. So $z(x)$ is well defined, and $u(x) \ge 0$ if and only if $z(x) \ge 0$. By the differentiability of $u(x)$ and $m(x)$, we have $z \in \mathcal{C}^2(d, b)$, $z \in \mathcal{C}[d, b]$ and

$$\begin{aligned} L(u) &= (m(x)z(x))'' + \alpha(x)\left(m(x)z(x)\right)' + \beta(x)(m(x)z(x) \\ &= m(x)z''(x) + \left[2m'(x) + \alpha(x)m(x)\right] z'(x) + L(m)z(x). \end{aligned}$$

Since $L(u) \le 0$ and $m > 0$, the above inequality is equivalent to

$$z''(x) + \frac{2m'(x) + \alpha(x)m(x)}{m(x)} z'(x) + \frac{L(m)}{m(x)} z(x) \le 0. \tag{2}$$

By the continuity of z on $[d,b]$, in order to prove $z(x) \geq 0$ we only need to show

$$\min_{x\in[d,b]} z(x) = \min\left\{ z(d),\ z(b),\ \min_{x\in(d,b)} z(x) \right\} \geq 0. \tag{3}$$

From $m(d) > 0, m(b) > 0$ and the assumptions $u(d) \geq 0, u(b) \geq 0$ we have

$$z(d) = \frac{u(d)}{m(d)} \geq 0, \qquad z(b) = \frac{u(b)}{m(b)} \geq 0.$$

If $z(c) = \min_{x\in(d,b)} z(x) < 0$ for some $c \in (d,b)$, then $z'(c) = 0, z''(c) \geq 0$. Evaluating inequality (2) at $x = c$,

$$z''(c) + \frac{L(m)(c)}{m(c)} z(c) \leq 0, \quad L(m)(c) > 0,\ m(c) > 0,\ z''(c) \geq 0 \quad \Longrightarrow \quad z(c) \leq 0$$

which would contradict $z(c) < 0$. Therefore

$$\min_{x\in(d,b)} z(x) \geq 0 \quad \Longrightarrow \quad (3)\ \textit{holds} \quad \Longrightarrow \quad u(x) = z(x)m(x) \geq 0,\ \forall x \in [d,b].$$

This completes the proof of Lemma 12. ☐

Now we prove the main result Theorem 1.

Proof of Theorem 1. Recall $L(u) = u'' + \alpha(x)u' + \beta(x)u \leq 0$, $|\alpha(x)|,\ |\beta(x)| < C$. Let

$$w(x) = u(x) - \epsilon h(x), \qquad x \in [a,b]$$

where $\epsilon > 0$ is to be determined, and let

$$h(x) = e^{\gamma(x-b)} - 1, \qquad \gamma = 2C + 1.$$

Notice that $h(b) = 0$, $L(h) > 0$ for $x \in (a,b)$ by Lemma 10. When $u \in \mathcal{C}(a,b]$, $u(b) = 0$, we have

$$w(b) = u(b) - \epsilon h(b) = 0, \qquad L(w) = L(u) - \epsilon L(h) \leq 0, \quad x \in (a,b).$$

By the assumption, there are $x_k \to b$ such that $u(x_k) > 0$. Since $h > 0$, we may choose k_0 and ϵ such that

$$b - x_{k_0} \leq \frac{\ln(2C+2)}{4C+2}, \qquad \epsilon = \frac{u(x_{k_0})}{h(x_{k_0})} > 0.$$

Then

$$w(x_{k_0}) = u(x_{k_0}) - \epsilon_0 h(x_{k_0}) = 0.$$

Applying Lemma 12 on $(d,b) = (x_{k_0}, b)$,

$$w(x_{k_o}) = w(b) = 0, \quad L(w) \leq 0,\ x \in (x_{k_0}, b) \qquad \Longrightarrow \quad w \geq 0,\ x \in (x_{k_0}, b).$$

So

$$w(x) = w(x) - w(b) \geq 0 \quad \Longrightarrow \quad u(x) - u(b) \geq \epsilon(h(x) - h(b)), \quad x \in (x_{k_0}, b).$$

Consequently,

$$\frac{u(x) - u(b)}{x - b} \leq \epsilon \frac{h(x) - h(b)}{x - b}, \qquad x \in (x_{k_0}, b).$$

Taking the limit $x_k \to b$,

$$\limsup_{x_k\to b} \frac{u(x_k)-u(b)}{x_k-b} \le -\epsilon h'(b) = -\epsilon\gamma < 0.$$

Thus

$$\limsup_{x\to b-} \frac{u(x)}{x-b} = \limsup_{x\to b-} \frac{u(x)-u(b)}{x-b} \le \limsup_{x_k\to b} \frac{u(x_k)-u(b)}{x_k-b} < 0.$$

If instead we have $u \in \mathcal{C}(a,b]$, $u(a) = 0$ and $\{x_k\} \subset (a,b)$ such that $x_k \to a+,\ u(x_k) > 0$, then an analogous argument will lead to

$$\limsup_{x\to a+} \frac{u(x)}{x-a} > 0.$$

This completes the proof of Theorem 1. ☐

Remarks on Theorem 1.

- The result of Theorem 1 includes the case $\limsup\limits_{x\to b-} \dfrac{u(x)}{x-b} = -\infty$.
- From the last step of the proof we can see that, under the assumption $u \in \mathcal{C}^1(a,b]$ in Theorem 1*, we have

  $$u'(b) = u'_-(b) \le \limsup_{x_k\to b} \frac{u(x_k)-u(b)}{x_k-b} < 0.$$

  Therefore Theorem 1* holds.

3. Proofs of corollaries and other theorems

The following is the proof of Corollary 2, which establishes the equivalence of Theorem 1 and the classical Hopf lemma.

Proof of Corollary 2. By the assumption, there is a sequence $\{x_k\} \subset (a,b)$ such that $x_k \to b,\ u(x_k) > 0$. We need to show $u(x) > 0$ in a neighborhood of b.

If $u(x) \ge 0$ but not $u(x) > 0$ in any neighborhood of b, there would exist a sequence $\{b_k\} \subset (a,b),\ b_k \to b$ such that $u(b_k) = 0$. Then the limit

$$\limsup_{b_k\to b} \frac{u(b_k)-u(b)}{b_k-b} \le \limsup_{x\to b-} \frac{u(x)-u(b)}{x-b} < 0$$

by the result of Theorem 1, contradicting to the fact that the left most term $\equiv 0$.

If $u(x) \ge 0$ did not hold in any neighborhood of b, there would exist a sequence $\{b_k\} \subset (a,b),\ b_k \to b$ such that $u(b_k) < 0$. By the assumption, there is a sequence $x_k \to b,\ u(x_k) > 0$, so $u(x)$ would oscillate around 0 in any neighborhood of b. By the convergence of $b_k,\ x_k$ to b and the continuity of u, there would exist k_0 and b', b'' such that $b' < b_{k_0} < b''$ and

$$u(b') = u(b'') = 0, \qquad b'' - b' < \frac{\ln(2C+2)}{4C+2}.$$

Since $L(u) \leq 0$, Lemma 12 $\Rightarrow u \geq 0$ on (b', b''), contradicting to $u(b_{k_0}) < 0$.

We have proved Corollary 2 by proof of contradiction. □

We now prove Theorem 4, which describes the behavior of zeros of $u(x)$ in $[a, b]$.

Proof of Theorem 4. By the definition of J_+, J_-, We my write

$$J_+(u) = \bigcup_{j=1}^{\infty} (a_j, b_j) \qquad \textit{with} \quad u(x) > 0 \quad \forall x \in (a_j, b_j), \quad \forall j,$$

$$J_-(u) = \bigcup_{j=1}^{\infty} (a'_j, b'_j) \qquad \textit{with} \quad u(x) < 0 \quad \forall x \in (a'_j, b'_j), \quad \forall j,$$

where we allow the interval to be half open if a or b is one of the endpoints. Since $u(x) < 0, \forall x \in (a'_j, b'_j) \subset J_-$ and $u(a'_j) = u(b'_j) = 0$ for any j, Lemma 12 and $L(u) \leq 0$ imply $b'_j - a'_j \geq \frac{\ln(2C+2)}{4C+2}$. Thus J_- contains finitely many intervals, and the number of intervals N_- satisfies

$$N_- \left(\frac{\ln(2C+2)}{4C+2} \right) \leq \sum_{j=1}^{N^-} (b'_j - a'_j) \leq b - a \qquad \Longrightarrow \qquad N_- \leq (b-a) \frac{4C+2}{\ln(2C+2)}.$$

Next we show that each interval in J_+ must have its neighbors in J_-. For any $(a_j, b_j) \in J_+$ and any sequence $x_k \in (a_j, b_j), x_k \to b_j$, by definition we must have $u(x_k) > 0$ and $u(b_j) = 0$. If $b_j < b$, then $u'(b_j) < 0$ by applying Theorem 1* to interval (a_j, b_j). Therefore u is strictly decreasing at b_j, so $u(x) < 0$ for $x > b_j$ in the immediate neighborhood of b_j, thus (a_j, b_j) must be the left neighbor of an interval $(a'_i, b'_i) = (b_j, b'_i) \subset J_-$ for some i. Similarly, if $a < a_j$, for any sequence $x_\ell \in (a_j, b_j), x_\ell \to a_j$, we must have $u(x_\ell) > 0, u(a_j) = 0$, hence $u'(a_j) > 0$ by the symmetric version of Theorem 1* (with respect to the left endpoint) applied on interval (a_j, b_j). Therefore u is strictly increasing at a_j, then (a_j, b_j) must be adjacent to an interval $(a'_i, b'_i) = (a'_i, a_j) \subset J_-$ for some i. Thus each interval in J_+ has its neighbors in J_-, hence J_+ contains N_+ intervals such that $|N_+ - N_-| \leq 1$.

The set $J_0 = \{x \in [a, b], f(x) = 0\}$ is compact by definition, so J_0 consists of closed intervals and discrete points. If $[a_0, b_0] \subset J_0$, then a_0 and b_0 are not transverse zeros. Thus a_0, b_0 are either a or b or the endpoints of some intervals in J_-, because the endpoints of intervals in J_+ can only be transverse zeros as proved above. Thus the intervals in J_0 are adjacent by intervals in J_-, so J_0 contains N_0 intervals and $|N_0 - N_-| \leq 1$.

By the property of transverse zeros, the set of end points $\{a_j, b_j, a'_j, b'_j\}$ of J_-, J_+ includes all transverse zeros of u. Thus the number of transverse zeros Z_0 is finite and $|Z_0 - N_- - N_+| \le 1$.

This concludes the proof of Theorem 4. ☐

Since

$$|Z_0 - N_- - N_+| \le 1 \quad \Rightarrow \quad Z_0 \le N_- + N_+ + 1 \le 2N_- + 2 \le 2(b-a)L_C,$$

we can see that Theorem 5 is a rephrase of parts of Theorem 4 so no new proof is needed. We now prove Corollary 6 which describes a type of convexity of u in Theorem 1.

Proof of Corollary 6. By the assumption, $u(x_0) = u'(x_0) = 0$. If $u \le 0$ did not hold in a neighborhood of x_0, there would exist a sequence $x_j \to x_0$ with $u(x_j) > 0$. We only need to consider either $x_j < x_0, \forall j$ or $x_j > x_0, \forall j$.

If $x_j < x_0, \forall j$, Theorem 1^* applies to $u(x)$ and $\{x_j\}$ on interval (a, x_0), thus by Theorem 1^*, $u'(x_0) < 0$, contradicting to the assumption $u'(x_0) = 0$.

If $x_j > x_0, \forall j$, the same conclusion can be arrived by using the symmetric version of Theorem 1^* (for the case $u(a) = 0, x_k \to a, u(x_k) > 0$) on interval (x_0, b).

This concludes the proof of Corollary 6. ☐

Below we prove Corollary 7 which is on general critical points.

Proof of Corollary 7. Let $\tilde{u}(x) = u(x) - u(x_0)$, then $\tilde{u}'(x_0) = \tilde{u}(x_0) = 0$. By the assumption, there is $\delta > 0$ such that $\beta(x)u(x_0) \ge 0$ for $x \in (x_0 - \delta, x_0 + \delta)$. Therefore,

$$L(\tilde{u})(x) = L(u)(x) - \beta(x)u(x_0) \le 0, \qquad x \in (x_0 - \delta, x_0 + \delta).$$

Applying Corollary 6 to $(x_0 - \delta, x_0 + \delta)$ we have $\tilde{u}(x) \le 0$ in a neighborhood of x_0, which implies $u(x) \le u(x_0)$ in a neighborhood of x_0. This concludes the proof of Corollary 7. ☐

Next we prove Theorem 8, which is an application of Theorem 1 and Corollary 6 to nonlinear functionals.

Proof of Theorem 8. Since $F \in \mathcal{C}^1(\mathbb{R}^4)$, we may write

$$\begin{aligned} 0 &\ge F(x, u, u', u'') - F(x, v, v', v'') \\ &= \int_0^1 \frac{\partial F}{\partial \theta}\big(x,\ \theta u + (1-\theta)v,\ \theta u' + (1-\theta)v',\ \theta u'' + (1-\theta)v''\big)\, d\theta \\ &= (u - v)\int_0^1 \frac{\partial F}{\partial x_2} d\theta \ +\ (u' - v')\int_0^1 \frac{\partial F}{\partial x_3} d\theta \ +\ (u'' - v'')\int_0^1 \frac{\partial F}{\partial x_4} d\theta \end{aligned}$$

Since $\frac{\partial F}{\partial x_4} > 0$, we may define

$$\alpha(x) = \int_0^1 \frac{\partial F}{\partial x_2} d\theta \Big/ \int_0^1 \frac{\partial F}{\partial x_4} d\theta, \qquad \beta(x) = \int_0^1 \frac{\partial F}{\partial x_3} d\theta \Big/ \int_0^1 \frac{\partial F}{\partial x_4} d\theta. \tag{4}$$

Let $w = u - v$ for $x \in (a, b)$. Then $w \in \mathcal{C}^2(a, b),\ w \in \mathcal{C}[a, b]$ and

$$w(x_0) = w'(x_0) = 0, \qquad L(w) = w'' + \alpha(x)w' + \beta(x)w \leq 0, \quad x \in (a, b).$$

Corollary 6 implies $w(x) \leq w(x_0) = 0$ in a neighborhood of x_0, thus $u(x) \leq v(x)$ in a neighborhood of x_0. This concludes the proof of Theorem 8. □

The beginning of the proof of Corollary 9 is analogous to that of Theorem 8, then Theorem 4 is applied.

Proof of Corollary 9. Follow the proof of Theorem 8, the assumptions lead to

$$L(w) = w'' + \alpha(x)w' + \beta(x)w \leq 0$$

where $w = u - v,\ w \in \mathcal{C}^2(a, b),\ w \in \mathcal{C}[a, b]$ and $\alpha(x), \beta(x)$ are as in (4). By Theorem 4, the number of transverse zeros of w is bounded by $2(b-a)L_C + 2$, and the number of identically zero intervals of w is bounded by $(b-a)L_C+1$. Therefore the number of transverse intersections of u, v is bounded by $2(b - a)L_C + 2$, and the number of intervals where $u \equiv v$ is bounded by $(b-a)L_C + 1$. This proves the conclusion of Corollary 9. □

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