Real Analysis Exchange

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AN APPLICATION OF HARDY-LITTLEWOOD TAUBERIAN THEOREM TO HARMONIC EXPANSION OF A COMPLEX MEASURE ON THE SPHERE

Abstract

We apply Hardy-Littlewood's Tauberian theorem to obtain an estimate on the harmonic expansion of a complex measure on the unit sphere, using a monotonicity property for positive harmonic functions.

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\}$, $n \geq 2$ be the unit ball in $\mathbb{R}^n$ and $S^{n-1} = \partial B^n$ be the unit sphere. From a monotonicity property, we obtain a precise asymptotic for the spherical harmonic expansion of a complex measure on S^{n-1} by applying the Tauberian theorem of Hardy and Littlewood.

It is known [1] that a positive harmonic function u in B^n can be uniquely represented by the Poisson kernel $P(x, y)$ and a positive measure μ on S^{n-1} as

$$u(x) = P[\mu](x) = \int_{S^{n-1}} P(x, \eta) d\mu(\eta) = \int_{S^{n-1}} \frac{1 - |x|^2}{|x - \eta|^n} d\mu(\eta). \qquad (1)$$

In the following we state a monotonicity property for positive harmonic functions as a theorem (Theorem 1), which is the special case $\delta = 0$ of Theorem 1.1 in [5]. A corollary (Corollary 2) on asymptotic results follow. Then we

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apply the monotonicity and the asymptotic property to obtain an estimate on the spherical harmonic expansion of a complex measure on S^{n-1} (Theorem 3) by applying Hardy-Littlewood's Tauberian Theorem. Two corollaries follow.

Theorem 1. *(Theorem 1.1 in [5]) Let u be a positive harmonic function in B^n, $\zeta \in S^{n-1}$. Then the function*

$$\frac{(1-r)^{n-1}}{1+r}u(r\zeta)$$

is decreasing and the function

$$\frac{(1+r)^{n-1}}{1-r}u(r\zeta)$$

is increasing for $0 \leq r < 1$.

The following is needed to prove our main result in Theorem 3.

Corollary 2. *Let u be a positive harmonic function in B^n defined by a positive measure μ as in (1). Then*

$$\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) = 2\mu(\{\zeta\})$$

and

$$\lim_{r\to 1}\frac{u(r\zeta)}{1-r} = \int_{S^{n-1}}\frac{2}{|\zeta-\eta|^n}d\mu(\eta).$$

PROOF. Applying Theorem 1 to the Poisson kernel we obtain

$$\frac{(1-r)^{n-1}}{1+r}P(r\zeta,\eta) = \frac{(1-r)^n}{|r\zeta-\eta|^n} \searrow \delta(\zeta,\eta) = \begin{cases} 1, & \zeta = \eta \\ 0, & \zeta \neq \eta \end{cases} \qquad as \quad r \to 1.$$

By the representation (1) and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) &= \lim_{r\to 1}(1-r)^{n-1}\int_{S^{n-1}}P(r\zeta,\eta)d\mu(\eta) \\ &= \lim_{r\to 1}(1+r)\int_{S^{n-1}}\lim_{r\to 1}\left\{\frac{(1-r)^{n-1}}{1+r}P(r\zeta,\eta)\right\}d\mu(\eta) \\ &= 2\mu(\{\zeta\}).\end{aligned}$$

Similarly, $\dfrac{(1+r)^{n-1}}{1-r}P(r\zeta,\eta) = \dfrac{(1+r)^n}{|r\zeta-\eta|^n}$ increases as $r \to 1$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}
\lim_{r\to 1}\frac{u(r\zeta)}{1-r} &= \lim_{r\to 1}\frac{1}{1-r}\int_{S^{n-1}} P(r\zeta,\eta)d\mu(\eta) \\
&= \lim_{r\to 1}\frac{1}{(1+r)^{n-1}}\int_{S^{n-1}} \lim_{r\to 1}\left\{\frac{(1+r)^{n-1}}{1-r}P(r\zeta,\eta)\right\} d\mu(\eta) \\
&= \frac{1}{2^{n-1}}\int_{S^n}\frac{2^n}{|\zeta-\eta|^n}d\mu(\eta) = \int_{S^n}\frac{2}{|\zeta-\eta|^n}d\mu(\eta).
\end{aligned}$$

□

Let $\mathcal{H}_m(S^{n-1})$ denote the complex vector space of spherical harmonics of degree m. $\mathcal{H}_m(S^{n-1})$ is the restriction to S^{n-1} of the complex vector space $\mathcal{H}_m(\mathbb{R}^n)$ of homogeneous harmonic polynomials of degree m in $\mathbb{R}^n$. It is known [1] that

$$\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1},$$

and that under the inner product $\langle p, q\rangle = \int_{S^{n-1}} p(x)q(x)d\sigma(x)$, where $d\sigma$ is the normalized Lebesgue measure on S^{n-1}, there exists an orthogonal decomposition of the Hilbert space of square-integrable functions on S^{n-1},

$$\mathcal{L}^2(S^{n-1}) = \oplus_0^\infty \mathcal{H}_m(S^{n-1}).$$

By the property of finite dimensional Hilbert space, $\forall \zeta \in S^{n-1}$, there exists a unique $Z_m(\zeta,\cdot) \in \mathcal{H}_m(S^{n-1})$ (the *zonal function* of pole ζ and order m) such that

$$p_m(\zeta) = \int_{S^{n-1}} p_m(\eta)Z_m(\zeta,\eta)d\sigma(\eta), \quad \forall p_m \in \mathcal{H}_m(S^{n-1}).$$

The above leads to a zonal expansion of the Poisson kernel (Theorem 5.33 in [1])

$$P(x,\zeta) = \frac{1-|x|^2}{|x-\zeta|^n} = \sum_{m=0}^\infty Z_m(x,\zeta), \qquad \forall x \in B^n,\ \zeta \in S^{n-1}. \tag{2}$$

Consequently, any complex measure on S^{n-1} has a spherical harmonic expansion

$$\sum_{m=0}^\infty p_m(\zeta), \qquad p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta,\eta)d\mu(\eta) \in \mathcal{H}_m(S^{n-1}),\ \zeta \in S^{n-1}.$$

If $f \in L^2(S^{n-1})$ and $d\mu = f d\sigma$, then the spherical harmonic expansion for μ converges to f in $\mathcal{L}^2(S^{n-1})$. It is known [3] that if $1 \leq p < 2, n > 2$ then there is an $\phi \in L^p(S^{n-1})$ with spherical harmonic expansion divergent almost everywhere. There have been studies of general theory of Cesàro summability on spherical harmonic expansions of L^p functions through estimates (e.g. [2]). In this paper, we consider spherical harmonic expansion of complex measures through asymptotics, which is the exact situation applicable by the Hardy-Littlewood Tauberian theory.

In the following we provide a precise asymptotics for the spherical harmonic expansion of complex measures on S^{n-1}.

Theorem 3. *Let μ be a complex Borel measure on the unit sphere S^{n-1} . Let $\sum_{m=0}^{\infty} p_m(\zeta)$ be the spherical harmonic expansion of μ. Then*

$$\sum_{m=0}^{N} p_m(\zeta) \;\sim\; \frac{2}{(n-1)!}\,\mu(\{\zeta\})\;N^{n-1} \qquad \text{as} \quad N \to \infty. \tag{3}$$

The proof of Theorem 3 is an application of the well-known Hardy-Littlewood Tauberian Theorem [4] stated below.

Hardy-Littlewood Tauberian Theorem. *Assume that $\sum_{m=0}^{\infty} a_m x^m$ converges on $|x| < 1$. Suppose that for some number $\alpha \geq 0$,*

$$\sum_{m=0}^{\infty} a_m x^m \sim \frac{A}{(1-x)^{\alpha}} \qquad \text{as} \quad x \nearrow 1$$

while

$$m a_m \geq -C m^{\alpha}, \quad m \geq 1,$$

then

$$\sum_{m=0}^{N} a_m \sim \frac{A}{\Gamma(\alpha+1)} N^{\alpha}.$$

Another known result crucial in our proof is stated below as Lemma 4, which is a modified version of Corollary 5.34 in [1].

Lemma 4. *Let μ be a complex measure on S^{n-1} and $u(x) = P[\mu](x)$ as in (1). Then there exist $p_m \in \mathcal{H}_m(\mathbb{R}^n)$, $m = 0, 1, 2, \cdots$ such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x), \qquad x \in B^n$$

and the series converges absolutely and uniformly on compact subsets of B^n. Furthermore, there is a positive constant C such that

$$|p_m(x)| \leq C|\mu(S^{n-1})|m^{n-2}|x|^m, \qquad m = 0, 1, 2, \cdots .$$

If $x = |x|\zeta$, then $p_m(\zeta)$ is given by

$$p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}).$$

PROOF. Our proof of Lemma 4 is a modified version of the proof of Corollary 5.34 in [1] in terms of measures. By Theorem 5.33 of [1], the Poisson kernel expansion by zonal harmonics (2) converges absolutely and uniformly on $K \times S^{n-1}$ for every compact set $K \subset B^n$. So for any $x \in B^n$,

$$u(x) = \int_{S^{n-1}} P(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} \int_{S^{n-1}} P(x, \zeta) Z_m(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} p_m(x)$$

where

$$p_m(x) = \int_{S^{n-1}} Z_m(x, \zeta) d\mu(\zeta), \qquad x \in B^n.$$

Since $p_m(x) \in \mathcal{H}_m(\mathbb{R}^n)$., so for $x = |x|\eta$, we have $Z_m(x, \zeta) = |x|^m Z_m(\eta, \zeta)$. Furthermore, it is known that [1]

$$|Z_m(\eta, \zeta)| \leq \dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$$

By Pascal's triangle,

$$\begin{aligned} \dim \mathcal{H}_m(\mathbb{R}^n) &= \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2} \\ &= \frac{1}{(n-2)!} \left(\frac{n+2m-2}{m} \right) \frac{(n+m-3)!}{(m-1)!}. \end{aligned}$$

Applying Stirling's formula,

$$\frac{\dim \mathcal{H}_m(\mathbb{R}^n)}{m^{n-2}} \to \frac{2}{(n-2)!} \qquad as \quad m \to \infty.$$

Therefore there exists $C = C(n) > 0$ such that $|Z_m(x,\zeta)| \leq Cm^{n-2}$ and

$$|p_m(x)| \leq \left| \int_{S^{n-1}} |Z_m(x,\zeta)| d\mu(\zeta) \right| \leq C|\mu(S^{n-1})| m^{n-2} |x|^m.$$

This completes the proof of Lemma 4. □

Below is the proof of our main result.

PROOF OF THEOREM 3. From the above results, for $x = r\zeta,\ \zeta \in S^{n-1}$, we can write

$$u(x) = P[\mu](x) = \sum_{m=0}^{\infty} p_m(x) = \sum_{m=0}^{\infty} p_m(\zeta) r^m,$$

and the last series converges for $|r| < 1$ by Lemma 4. The complex Borel measure μ can be decomposed as

$$\mu = \mathrm{Re}(\mu) + i\ \mathrm{IM}(\mu) = (\mu_1 - \mu_2) + i(\mu_3 - \mu_4)$$

where $\mu_j,\ j = 1,2,3,4$ are positive Borel measures. Applying Lemma 4 and Corollary 2 to the μ_j's and combining the resulting expansions, we have

$$\sum_{m=0}^{\infty} p_m(\zeta) r^m \sim \frac{2\mu(\{\zeta\})}{(1-r)^{n-1}} \qquad \text{as} \quad r \nearrow 1$$

Taking real and imaginary parts we have

$$\sum_{m=0}^{\infty} \mathrm{Re}\{p_m(\zeta)\} r^m \sim \frac{2\ \mathrm{Re}\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \qquad \text{as} \quad r \nearrow 1$$

and

$$\sum_{m=0}^{\infty} \mathrm{Im}\{p_m(\zeta)\} r^m \sim \frac{2\ \mathrm{Im}\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \qquad \text{as} \quad r \nearrow 1.$$

By Lemma 4, there exists a positive constant C so that

$$|p_m(\zeta)| \leq Cm^{n-2}$$

It follows that

$$m\ \mathrm{Re}\,\{p_m(\zeta)\} \geq -Cm^{n-1}, \qquad m\ \mathrm{Im}\,\{p_m(\zeta)\} \geq -Cm^{n-1}.$$

Applying Hardy-Littlewood Tauberian Theorem with $\alpha = n-1$ we obtain (3). This completes the proof of Theorem 3. □

Corollary 5. *Let μ be a complex Borel measure on S^{n-1}. If $\mu(\{\zeta\}) > 0$ for some $\zeta \in S^{n-1}$ then the spherical expansion series of μ is divergent:*

$$\sum_{m=0}^{\infty} p_m(\zeta) = +\infty.$$

If $\mu(\{\zeta\}) = 0$ then

$$\sum_{m=0}^{N} p_m(\zeta) = o(N^{n-1}).$$

When the dimension of the space $n = 2$, the spherical expansion corresponds to Fourier series, and Theorem 3 has the following form which is a well known classical result.

Corollary 6. *Let μ be a complex Borel measure on S^1. Let*

$$\sum_{m=-\infty}^{\infty} a_m e^{im\theta}, \qquad a_m = \int_{-\pi}^{\pi} e^{-im\theta} d\mu(e^{i\theta})$$

be the Fourier series of μ. Then

$$\sum_{m=-N}^{N} a_m e^{im\theta} \sim 2\mu(\{e^{i\theta}\})N \qquad \text{as} \quad N \to \infty.$$

PROOF. In $\mathbb{R}^2$ the zonal functions are given by

$$Z_m(e^{i\theta}, e^{i\phi}) = e^{im(\theta-\phi)} + e^{-im(\theta-\phi)}$$

for $m > 0$, and $Z_0(e^{i\theta}, e^{i\phi}) = 1$. So Corollary 6 follows from Theorem 3. □

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References

[1] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory*, 2nd ed., Springer, New York (2001).

[2] A. Bonami, J-L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques*, Trans. Amer. Math. Soc. **183** (1973), 223-263.

[3] C. Kenig, R. Stanton, P. Tomas, *Divergence of eigenfunction expansions*, J. Funct. Anal. **46** (1982) , 28-44.

[4] J. Korevaar, *Tauberian Theory: a century of developments*, Springer (2004) .

[5] Y. Pan, M. Wang, *Monotonicity of Harnack inequality for positive invariant harmonic functions*, Appl. Math. Stoc. Anal. (2007), doi:10.1155/2007/39171.