STUDY OF HARMONIC FUNCTIONS AND MEASURES

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Contents

1. On monotonicity of positive invariant harmonic functions

A monotonicity property of Harnack inequality is proved for positive invariant harmonic functions in the unit ball.

1.1. **Statements of results.**

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\},\ n \geq 2$ be the open unit ball in $\mathbb{R}^n$. $S^{n-1} = \partial B^n$. Consider the differential operator

$$\Delta_\lambda = (1-|x|^2)\left\{\frac{1-|x|^2}{4}\sum_j \frac{\partial^2}{\partial x_j^2} + \lambda \sum_j x_j \frac{\partial}{\partial x_j} + \lambda(\frac{n}{2}-1-\lambda)\right\},\ \lambda \in \mathbb{R}.$$

We prove a monotonicity property of invariant harmonic functions that are solutions of $\Delta_\lambda u = 0$ and are defined by positive Borel measures on the sphere with respect to the Poisson kernel P_λ (see below).

This section describes the theorems and their corollaries. The proofs are provided in the next two sections.

Theorem 1.1. *Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. For $\zeta \in S^{n-1}$, if $\lambda > -\frac{n}{2}$ (if $\lambda < -\frac{n}{2}$), the function*

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}} u(r\zeta)$$

is decreasing (increasing) for $0 \leq r < 1$, and the function

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} u(r\zeta)$$

is increasing (decreasing) for $0 \leq r < 1$. Also

$$\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) = \begin{cases} 2^{1+2\lambda}\mu(\{\zeta\}), & \lambda > -\frac{n}{2} \\ \infty, & \lambda < -\frac{n}{2},\ \mu(\{\zeta\}^c) > 0 \\ 2^{1+2\lambda}\mu(\{\zeta\}), & \lambda < -\frac{n}{2},\ \mu(\{\zeta\}^c) = 0 \end{cases}$$

and

$$\lim_{r\to 1}\frac{u(r\zeta)}{(1-r)^{1+2\lambda}} = \int_{S^{n-1}} \frac{2^{1+2\lambda}}{|\zeta-\xi|^{n+2\lambda}} d\mu(\xi).$$

Remarks.

(1) Invariant harmonic functions are the solutions of $\Delta_\lambda u = 0$. These solutions also satisfy certain invariance property with respect to Möbius transformation. Invariant harmonic functions generally do not possess good boundary regularity, as shown in Liu and Peng [11].

(2) Let μ be a positive Borel measure on S^{n-1} and P_λ be the Poisson kernel

$$P_\lambda(x,\zeta) = \frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}.$$

It is known that the integral

$$u(x) = \int_{S^{n-1}} P_\lambda(x,\zeta)d\mu(\zeta)$$

defines an invariant harmonic function in B^n ([1], p. 119).

(3) On the completion of the current work, we learned that the limit cases for $n = 2, \lambda = 0$ in Theorem 1.1 were obtained by Simon and Wolff ([18], ref. Chapter 10, p. 546 in [17]).

(4) The critical value $\lambda = -\frac{n}{2}$ yields the degenerate case with the constant Poisson kernel.

The following theorem characterizes the behavior of invariant harmonic functions on the rays.

Theorem 1.2. *Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. Let $\zeta \in S^{n-1}$ and $0 \le r' \le r < 1$.*

If $\lambda > -\frac{n}{2}$,

$$\left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1}u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1}u(r'\zeta)$$

If $\lambda < -\frac{n}{2}$,

$$\left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1}u(r'\zeta) \le u(r\zeta) \le \left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1}u(r'\zeta)$$

For $r' = 0$, the above becomes

$$\frac{(1-r)^{1+2\lambda}}{(1+r)^{n-1}}u(0) \le u(r\zeta) \le \frac{(1+r)^{1+2\lambda}}{(1-r)^{n-1}}u(0)$$

for $\lambda > -\frac{n}{2}$, and

$$\frac{(1+r)^{1+2\lambda}}{(1-r)^{n-1}}u(0) \le u(r\zeta) \le \frac{(1-r)^{1+2\lambda}}{(1+r)^{n-1}}u(0)$$

for $\lambda < -\frac{n}{2}$.

Remark. Case $\lambda = 0$ is the classical Harnack Inequality in B^n.

Corollary 1.3. *Let U be the potential function defined in B^n by a positive Borel measure μ on S^{n-1} as follows:*

$$U(x) = \int_{S^{n-1}} \frac{1}{|x-\eta|^{n+2\lambda}}d\mu(\eta).$$

For $\zeta \in S^{n-1}$, if $\lambda > -\frac{n}{2}$ (if $\lambda < -\frac{n}{2}$), the function

$$(1-r)^{n+2\lambda}U(r\zeta)$$

is decreasing (increasing) for $0 \leq r < 1$.

In Theorem 1.1, $\lambda = \frac{n}{2} - 1$ corresponds to the Laplace-Beltrami operator $\Delta_{n/2-1}$ and the Poincaré metric. It is known([2]) that given a positive invariant harmonic function (solutions of $\Delta_{n/2-1}u = 0$), there exists a positive Borel measure μ on S^{n-1}, such that

$$u(x) = \int_{S^{n-1}} P_{n/2-1}(x, \zeta) d\mu(\zeta)$$

In this case the monotonicity property in Theorem 1.1 implies the following corollary.

Corollary 1.4. *Let* u *be a positive solution of* $\Delta_{n/2-1}u = 0$ *in* B^n. *Then*

$$\left(\frac{1-r}{1+r}\right)^{n-1} u(r\zeta) \text{ decreasing in } r,$$

$$\left(\frac{1+r}{1-r}\right)^{n-1} u(r\zeta) \text{ increasing in } r.$$

Corollary 1.5. *Let* u *be a positive harmonic function with respect to the Laplace operator* ($\lambda = 0$) *defined in* B^n *by a positive Borel measure* μ *on* S^{n-1} *with the Poisson Kernel* P_0. *For* $\zeta \in S^{n-1}$, $0 \leq r < 1$, *the function*

$$\frac{(1-r)^{n-1}}{1+r} u(r\zeta)$$

is decreasing and the function

$$\frac{(1+r)^{n-1}}{1-r} u(r\zeta)$$

is increasing. In addition,

$$\lim_{r \to 1} (1-r)^{n-1} u(r\zeta) = 2\mu(\{\zeta\}),$$

$$\lim_{r \to 1} \frac{u(r\zeta)}{1-r} = \int_{S^{n-1}} \frac{2}{|\zeta - \xi|^n} d\mu(\xi).$$

Corollary 1.5 is the same as a result in [12].

Corollary 1.6. *Let* $B^n(R)$ *be the open ball of radius* R. *Let* $u(z)$ *be an invariant harmonic function in* $B^n(R)$ (a.k.a $u(Rz)$ *is invariant harmonic in* B^n) *defined by the Poisson kernel* $P_\lambda(\frac{x}{R}, \zeta)$. *For* $\zeta \in S^{n-1}$, *if* $\lambda > -\frac{n}{2}$ (*if* $\lambda < -\frac{n}{2}$), *the function*

$$\frac{1}{R^{n-2-2\lambda}} \frac{(R-r)^{n-1}}{(R+r)^{1+2\lambda}} u(r\zeta)$$

is decreasing (increasing) and the function

$$\frac{1}{R^{n-2-2\lambda}} \frac{(R+r)^{n-1}}{(R-r)^{1+2\lambda}} u(r\zeta)$$

is increasing (decreasing) in r for $0 \leq r < R$. The case $\lambda = 0$ gives the monotonicity of functions

$$\left(\frac{R-r}{R}\right)^{n-2} \frac{R-r}{R+r} u(r\zeta) \qquad \text{and} \qquad \left(\frac{R+r}{R}\right)^{n-2} \frac{R+r}{R-r} u(r\zeta),$$

which implies that, $\forall x \in B^n(r), 0 \leq r < R$,

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r} u(0) \leq u(x) \leq \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r} u(0)$$

— the classical Harnack Inequality.

Corollary 1.7. *Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. Let $0 \leq r' \leq r < 1$. If $\lambda > -\frac{n}{2}$,*

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}} \max_{|x|=r} u(x) \leq \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}} \max_{|x|=r'} u(x)$$

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} \min_{|x|=r} u(x) \geq \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}} \min_{|x|=r'} u(x)$$

If $\lambda < -\frac{n}{2}$,

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} \max_{|x|=r} u(x) \leq \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}} \max_{|x|=r'} u(x)$$

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}} \min_{|x|=r} u(x) \geq \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}} \min_{|x|=r'} u(x)$$

Similar results are obtained in complex space $\mathbb{C}^n$. Let

$$P_\alpha(z,\zeta) = \frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}, \ \alpha \in \mathbb{R}$$

be the Poisson-Szegö kernel for the operator

$$\Delta_{\alpha,\beta} = 4(1-|z|^2)\left\{\sum_{i,j}(\delta_{i,j} - z_i\overline{z}_j)\frac{\partial^2}{\partial z_i \partial \overline{z}_j} + \alpha \sum_j z_j \frac{\partial}{\partial z_j} + \beta \sum_j \overline{z}_j \frac{\partial}{\partial \overline{z}_j} - \alpha\beta\right\}$$

with $\alpha = \beta$, where $z \cdot \overline{\zeta} = \sum_{i=1}^n z_i \overline{\zeta}_i$. Define

$$u(z) = \int_{S^{n-1}} P_\alpha(z,\zeta) d\mu(\zeta), \quad \alpha \in \mathbb{R}.$$

Theorem 1.8. *Let u be a positive invariant harmonic function defined in the unit ball $B^n \subset \mathbb{C}^n$ by a positive Borel measure μ on $S^{n-1} = \partial B^n$ with the Poisson-Szegö kernel. Given $\zeta \in S^{n-1}$, if $\alpha > -n$ (if $\alpha < -n$), the function*

$$\frac{(1-r)^n}{(1+r)^{n+2\alpha}} u(r\zeta)$$

is decreasing (increasing) for $0 \le r < 1$, and the function

$$\frac{(1+r)^n}{(1-r)^{n+2\alpha}} u(r\zeta)$$

is increasing (decreasing) for $0 \le r < 1$. Also

$$\lim_{r\to 1}(1-r)^n u(r\zeta) = \begin{cases} 2^{n+2\alpha}\mu(\{\zeta\}), & \alpha > -n \\ \infty, & \alpha < -n,\ \mu(\{\zeta\}^c) > 0 \\ 2^{n+2\alpha}\mu(\{\zeta\}), & \alpha < -n,\ \mu(\{\zeta\}^c) = 0 \end{cases}$$

and

$$\lim_{r\to 1} \frac{u(r\zeta)}{(1-r)^{n+2\alpha}} = \int_{S^{n-1}} \frac{2^{n+2\alpha}}{|\zeta-\eta|^{2n+2\alpha}} d\mu(\eta).$$

The following theorem describes invariant harmonic functions on the rays.

Theorem 1.9. *Let u be a positive invariant harmonic function defined in the unit ball $B^n \subset \mathbb{C}^n$ by a positive Borel measure μ on S^{n-1} with the Poisson-Szegö kernel. Let $\zeta \in S^{n-1}$ and $0 \le r' \le r < 1$.*

if $\alpha > -n$,

$$\left(\frac{1-r}{1-r'}\right)^{n+2\alpha} \left(\frac{1+r'}{1+r}\right)^n u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{n+2\alpha} \left(\frac{1-r'}{1-r}\right)^n u(r'\zeta)$$

If $\alpha < -n$,

$$\left(\frac{1+r}{1+r'}\right)^{-2n-2\alpha} \left(\frac{1-r^2}{1-r'^2}\right)^{n+2\alpha} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1-r}{1-r'}\right)^{n+2\alpha} \left(\frac{1+r'}{1+r}\right)^n u(r'\zeta)$$

For $r' = 0$, the above becomes

$$\frac{(1-r)^{n+2\alpha}}{(1+r)^n} u(0) \le u(r\zeta) \le \frac{(1+r)^{n+2\alpha}}{(1-r)^n} u(0)$$

for $\alpha > -n$, and

$$\frac{(1+r)^{n+2\alpha}}{(1-r)^n} u(0) \le u(r\zeta) \le \frac{(1-r)^{n+2\alpha}}{(1+r)^n} u(0)$$

for $\alpha < -n$.

1.2. **Proofs of Theorem 1.1 and its corollaries.** We need the following two lemmas for the proof of Theorem 1.1.

Lemma 1.10. *Let $x \in \mathbb{R}^n$, $|x| = r$, $\zeta \in S^{n-1}$.*

If $\lambda > -\frac{n}{2}$ then

$$-\frac{(n+2\lambda-(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \le \frac{\partial}{\partial r}\frac{(1-r^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} \le \frac{(n+2\lambda+(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \tag{1.1}$$

If $\lambda < -\frac{n}{2}$, then

$$\frac{(n+2\lambda+(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \le \frac{\partial}{\partial r}\frac{(1-r^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} \le -\frac{(n+2\lambda-(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \tag{1.2}$$

Proof. Write $x = |x|\eta = r\eta$, $\eta \cdot \zeta = \sum_{i=1}^n \eta_i\zeta_i$.

$$\frac{\partial}{\partial r}|x-\zeta|^2 = \frac{\partial}{\partial r}(|x|^2 - 2r\eta\cdot\zeta + 1) = 2(r - \eta\cdot\zeta),$$

then

$$\begin{aligned}\frac{\partial}{\partial r}|x-\zeta|^{n+2\lambda} &= \frac{\partial}{\partial r}(|x-\zeta|^2)^{\frac{n+2\lambda}{2}} \\ &= \frac{n+2\lambda}{2}(|x-\zeta|^2)^{\frac{n+2\lambda}{2}-1}\frac{\partial}{\partial r}|x-\zeta|^2 \\ &= (n+2\lambda)\ |x-\zeta|^{n+2\lambda-2}\ (r-\eta\cdot\zeta),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial r}\frac{(1-r^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} \\ &= \frac{(1+2\lambda)(1-r^2)^{2\lambda}(-2r)|x-\zeta|^{n+2\lambda} - (1-r^2)^{1+2\lambda}\frac{\partial}{\partial r}|x-\zeta|^{n+2\lambda}}{|x-\zeta|^{2(n+2\lambda)}} \\ &= \frac{-2(1+2\lambda)(1-r^2)^{2\lambda}r|x-\zeta|^{n+2\lambda} - (1-r^2)^{1+2\lambda}(n+2\lambda)|x-\zeta|^{n+2\lambda-2}(r-\eta\cdot\zeta)}{|x-\zeta|^{2(n+2\lambda)}} \\ &= \frac{-2(1+2\lambda)(1-r^2)^{2\lambda}r|x-\zeta|^2 - (1-r^2)^{1+2\lambda}(n+2\lambda)(r-\eta\cdot\zeta)}{|x-\zeta|^{n+2\lambda+2}}.\end{aligned} \tag{1.3}$$

To prove the right side inequality in (1.1), it suffices to show

$$-2(1+2\lambda)r|x-\zeta|^2 - (1-r^2)(n+2\lambda)(r-\eta\cdot\zeta) \le (n+2\lambda+(n-2\lambda-2)r)|x-\zeta|^2,$$

which is equivalent to

$$-(n+2\lambda)(1-r^2)(r-\eta\cdot\zeta) \le (n+2\lambda)(1+r)|x-\zeta|^2.$$

For $\lambda > -\frac{n}{2}$, the above becomes

$$-(1-r^2)(r-\eta\cdot\zeta) \le (1+r)|x-\zeta|^2,$$

or

$$-(1-r)(r-\eta\cdot\zeta) \le r^2 - 2\eta\cdot\zeta + 1,$$

which, after a simple simplification, is equivalent to

$$\eta \cdot \zeta \leq 1$$

The inequality is true since $\zeta, \eta \in S^{n-1}$. To prove the left side inequality in (1.1), it suffices to show (using the result of (1.3))

$$-2(1+2\lambda)r|x-\zeta|^2 - (1-r^2)(n+2\lambda)(r-\eta\cdot\zeta) \geq -(n+2\lambda-(n-2\lambda-2)r)|x-\zeta|^2,$$

which is equivalent to

$$(n+2\lambda)(1-r^2)(r-\eta\cdot\zeta) \leq (n+2\lambda)(1-r)|x-\zeta|^2.$$

For $\lambda > -\frac{n}{2}$, the inequality is equivalent to

$$(1-r^2)(r-\eta\cdot\zeta) \leq (1-r)|x-\zeta|^2,$$

which is, after a simplification,

$$-\eta\cdot\zeta \leq 1,$$

true since $\zeta, \eta \in S^{n-1}$. The proof of (1.2) for $\lambda < -\frac{n}{2}$ is parallel. This completes the proof of Lemma 1.10. □

Lemma 1.11. *Let u be a positive invariant harmonic function in B^n defined by a positive Borel measure on S^{n-1} with the Poisson kernel.*

If $\lambda > -\frac{n}{2}$,

$$-\frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}u(x) \leq \frac{\partial u(x)}{\partial r} \leq \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}u(x). \tag{1.4}$$

If $\lambda < -\frac{n}{2}$,

$$\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}u(x) \leq \frac{\partial u(x)}{\partial r} \leq -\frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}u(x). \tag{1.5}$$

Proof. By the Poisson integral representation of u in B^n,

$$u(x) = \int_{S^{n-1}} \frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta)$$

for a positive Borel measure μ. By (1.1) in Lemma 2.4 and μ being a positive measure,

$$\begin{aligned}
\int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\right) d\mu(\zeta) &\leq \int_{S^{n-1}} \frac{(n+2\lambda+(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta) \\
&= \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} \int_{S^{n-1}} \frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta) \\
&= \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}\ u(x)
\end{aligned}$$

when $\lambda > -\frac{n}{2}$. It follows that

$$\frac{\partial u(x)}{\partial r} = \int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\right) d\mu(\zeta) \leq \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}\, u(x).$$

The left side inequality in (1.4) can be proved in the same manner. For the equality case, consider $u_y(x) = u(x,y) = \dfrac{(1-|x|^2)^{1+2\lambda}}{|x-y|^{n+2\lambda}}$ which is invariant harmonic in $\mathbb{R}^n \setminus \{y\}$ for $y \in S^{n-1}$. A simple calculation shows that the equalities hold for $u_y(x)$ when $x = |x|y$ and $x = -|x|y$ respectively. The proof of (1.5) is similar. This completes the proof of Lemma 1.11. $\square$

Now we prove Theorem 1.1.

Proof. Consider $\varphi(r) = \dfrac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}$ and $\psi(r) = \dfrac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}$ for $0 \leq r < 1$.

$$\frac{\varphi'}{\varphi} = -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}\,,$$

$$\frac{\psi'}{\psi} = \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}\,.$$

Given $\omega \in S^{n-1}$, consider

$$\begin{aligned} I(r,\omega) &= \varphi(r)u(r\omega), \\ J(r,\omega) &= \psi(r)u(r\omega). \end{aligned}$$

To show Theorem 1.1, it suffices to show that $I(r,\omega)$ is decreasing (increasing) and $J(r,\omega)$ is increasing (decreasing) in r for $0 \leq r < 1$ when $\lambda > -\frac{n}{2}$ (when $\lambda < -\frac{n}{2}$). By (1.4) in Lemma 1.11, for $\lambda > -\frac{n}{2}$,

$$\begin{aligned} \frac{d}{dr}(\log I(r,\omega)) &= \frac{\varphi'}{\varphi} + \frac{u_r'}{u} \\ &= -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} + \frac{u_r'}{u} \\ &\leq -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} + \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} \\ &= 0. \end{aligned}$$

Therefore $\log I(r,\omega)$ is decreasing in r, and so is $I(r,\omega)$. Similarly,

$$\begin{aligned} \frac{d}{dr}(\log J(r,\omega)) &= \frac{\psi'}{\psi} + \frac{u_r'}{u} \\ &= \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} + \frac{u_r'}{u} \\ &\geq \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} - \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} \\ &= 0. \end{aligned}$$

Hence, $J(r,\omega)$ is increasing in r. For $\lambda > -\frac{n}{2}$ and $y \in S^{n-1}$,

$$\begin{aligned}\lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,y) &= \lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\frac{(1-|r|^2)^{1+2\lambda}}{|r\zeta-y|^{n+2\lambda}}\\ &= \lim_{r\to 1}\frac{(1-r)^{n+2\lambda}}{|r\zeta-y|^{n+2\lambda}} \searrow \delta(\zeta,y)=\begin{cases}1, & \zeta=y\\ 0, & \zeta\neq y\end{cases}\end{aligned}$$

by applying the monotonicity properties in Theorem 1.1 to $u = P_\lambda$. By Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) &= \lim_{r\to 1}(1-r)^{n-1}\int_{S^{n-1}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to 1}(1+r)^{1+2\lambda}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= 2^{1+2\lambda}\mu(\{\zeta\}).\end{aligned}$$

Similarly, $\dfrac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,y) = \dfrac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}$ is increasing in r for $\lambda > -\frac{n}{2}$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}\lim_{r\to 1}\frac{u(r\zeta)}{(1-r)^{1+2\lambda}} &= \lim_{r\to 1}\frac{1}{(1-r)^{1+2\lambda}}\int_{S^{n-1}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to 1}\frac{1}{(1+r)^{n-1}}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \frac{1}{2^{n-1}}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}d\mu(\xi)\\ &= \int_{S^{n-1}}\frac{2^{1+2\lambda}}{|\zeta-\xi|^{n+2\lambda}}d\mu(\xi).\end{aligned}$$

For $\lambda < -\frac{n}{2}$, the monotonicity of $I(r,\omega)$ and $J(r,\omega)$ is proved similarly to the case $\lambda > -\frac{n}{2}$ using (1.5) instead of (1.4) in Lemma 1.11.

$$\lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,y) = \lim_{r\to 1}\frac{|r\zeta-y|^{-(n+2\lambda)}}{(1-r)^{-(n+2\lambda)}} \nearrow \begin{cases}1, & \zeta=y\\ \infty, & \zeta\neq y\end{cases}$$

when $\lambda < -\frac{n}{2}$. Therefore,

$$\begin{aligned}\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) &= \lim_{r\to 1}(1+r)^{1+2\lambda}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \begin{cases}2^{1+2\lambda}\mu(\{\zeta\}), & if\ \mu(\{\zeta\}^c)=0;\\ \infty, & if\ \mu(\{\zeta\}^c)>0.\end{cases}\end{aligned}$$

Similarly, $\dfrac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,y)=\dfrac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}$ is decreasing in r for $\lambda<-\frac{n}{2}$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}\lim_{r\to1}\frac{u(r\zeta)}{(1-r)^{1+2\lambda}} &= \lim_{r\to1}\frac{1}{(1-r)^{1+2\lambda}}\int_{S^{n-1}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to1}\frac{1}{(1+r)^{n-1}}\int_{S_{n-1}}\lim_{r\to1}\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \frac{1}{2^{n-1}}\int_{S^{n-1}}\lim_{r\to1}\frac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}d\mu(\xi)\\ &= \int_{S^{n-1}}\frac{2^{1+2\lambda}}{|\zeta-\xi|^{n+2\lambda}}d\mu(\xi)\end{aligned}$$

This completes the proof of Theorem 1.1. □

The proof of Corollary 1.2 is straightforward and is omitted. The proof of Corollary 1.3 follows.

Proof.

$$\begin{aligned}(1-r)^{n+2\lambda}U(r\zeta) &= \int_{S^{n-1}}\frac{(1-r)^{n+2\lambda}}{|r\zeta-\eta|^{n+2\lambda}}d\mu(\eta)\\ &= \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\int_{S^{n-1}}\frac{(1-r^2)^{1+2\lambda}}{|r\zeta-\eta|^{n+2\lambda}}d\mu(\eta)\\ &= \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}u(r\zeta)\end{aligned}$$

which is decreasing (increasing) in r for $\lambda>-\frac{n}{2}$ ($\lambda<-\frac{n}{2}$) by Theorem 1.1. This completes the proof of Corollary 1.3. □

Corollaries 1.4 and 1.5 are special cases of Theorem 1.1. Corollary 1.6 is a straightforward generalization from B^n to $B^n(R)$. The following is the proof of Corollary 1.7.

Proof. $0\le r'\le r<1$. By the maximum principle, there is $\zeta\in S^{n-1}$ such that $u(r\zeta)=\max_{|x|=r}u(x)$.

If $\lambda>-\frac{n}{2}$, Theorem 1.1 implies

$$\begin{aligned}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\max_{|x|=r}u(x) &= \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}u(r\zeta)\\ &\le \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}}u(r'\zeta)\le\frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}}\max_{|x|=r'}u(x)\end{aligned}$$

Similarly, there is $\xi \in S^{n-1}$ such that $u(r\xi) = \min_{|x|=r} u(x)$. When $\lambda > -\frac{n}{2}$, Theorem 1.1 yields

$$\begin{aligned}\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} \min_{|x|=r} u(x) &= \frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} u(r\xi) \\ &\geq \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}} u(r'\xi) \geq \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}} \min_{|x|=r'} u(x)\end{aligned}$$

The proof for $\lambda < -\frac{n}{2}$ is parallel. This completes the proof of Corollary 1.7.

□

1.3. **Proof of Theorem 1.8.** In the following, B^n denotes the unit ball in $\mathbb{C}^n$ and $S^{n-1} = \partial B^n$ the sphere. We need the following three lemmas for the proof of Theorem 1.8.

Lemma 1.12. *If $a \in \mathbb{C}, |a| \leq 1$, then for $0 \leq r \leq 1$,*

$$1 + r|a|^2 \geq (1+r)\mathrm{Re}(a) \tag{1.6}$$

$$1 - r|a|^2 \geq (-1+r)\mathrm{Re}(a) \tag{1.7}$$

Proof. $|a| \leq 1$, so $-1 \leq -|a| \leq \mathrm{Re}(a) \leq |a| \leq 1$ and $\mathrm{Re}(a)^2 \leq |a|^2$.

If $|a|^2 \geq \mathrm{Re}(a)$, then $1 + r|a|^2 \geq \mathrm{Re}(a) + r\mathrm{Re}(a)$ so (1.6) holds.

If $|a|^2 < \mathrm{Re}(a)$, consider $f(r) = 1+r|a|^2-(1+r)\mathrm{Re}(a)$, $f'(r) = |a|^2-\mathrm{Re}(a) < 0$. So $f(r)$ decreases in $r \in [0,1]$. $f(1) = 1 + |a|^2 - 2\mathrm{Re}(a) > 1 + \mathrm{Re}(a)^2 - 2\mathrm{Re}(a) = (1 - Re(a))^2 \geq 0$. So $f(r) \geq 0$ and (1.6) holds.

For the second inequality, $1+\mathrm{Re}(a) \geq |a|^2+\mathrm{Re}(a) \geq r|a|^2+r\mathrm{Re}(a)$, so $1-r|a|^2 \geq (-1+r)\mathrm{Re}(a)$ and (1.7) holds. □

Lemma 1.13. *Let $z \in B^n$, $|z| = r$, $\zeta \in S^{n-1}$.*

If $\alpha > -n$ then

$$-\frac{(2n+2\alpha+2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \leq \frac{\partial}{\partial r}\frac{(1-r^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} \leq \frac{(2n+2\alpha-2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \tag{1.8}$$

If $\alpha < -n$, then

$$\frac{(2n+2\alpha-2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \leq \frac{\partial}{\partial r}\frac{(1-r^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} \leq -\frac{(2n+2\alpha+2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \tag{1.9}$$

Proof. Let $z = |z|\eta = r\eta$.

$$\frac{\partial}{\partial r}|1-z\cdot\overline{\zeta}|^2 = \frac{\partial}{\partial r}(1-2r\mathrm{Re}(\eta\cdot\overline{\zeta}) + r^2|\eta\cdot\overline{\zeta}|^2) = 2(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})),$$

and

$$\frac{\partial}{\partial r}|1 - z\cdot\overline{\zeta}|^{2n+2\alpha} = \frac{\partial}{\partial r}(|1 - z\cdot\overline{\zeta}|^2)^{n+\alpha}$$
$$= (n+\alpha)|(1 - z\cdot\overline{\zeta}|^2)^{n+\alpha-1}\frac{\partial}{\partial r}|1 - z\cdot\overline{\zeta}|^2$$
$$= 2(n+\alpha)|1 - z\cdot\overline{\zeta}|^{2n+2\alpha-2}(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})).$$

We have

$$\begin{aligned}\frac{\partial}{\partial r}\frac{(1-r^2)^{n+2\alpha}}{|1 - z\cdot\overline{\zeta}|^{2n+2\alpha}} &= \frac{(n+2\alpha)(1-r^2)^{n+2\alpha-1}(-2r)|1 - z\cdot\overline{\zeta}|^{2n+2\alpha}}{|1 - z\cdot\overline{\zeta}|^{4n+4\alpha}} \\ &- \frac{(1-r^2)^{n+2\alpha}\frac{\partial}{\partial r}|1 - z\cdot\overline{\zeta}|^{2n+2\alpha}}{|1 - z\cdot\overline{\zeta}|^{4n+4\alpha}} \\ &= \frac{-2(n+2\alpha)(1-r^2)^{n+2\alpha-1}r|1 - z\cdot\overline{\zeta}|^{2n+2\alpha}}{|1 - z\cdot\overline{\zeta}|^{4n+4\alpha}} \\ &- \frac{(1-r^2)^{n+2\alpha}2(n+\alpha)|1 - z\cdot\overline{\zeta}|^{2n+2\alpha-2}(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta}))}{|1 - z\cdot\overline{\zeta}|^{4n+4\alpha}} \\ &= \frac{-2(n+2\alpha)(1-r^2)^{n+2\alpha-1}r|1 - z\cdot\overline{\zeta}|^2}{|1 - z\cdot\overline{\zeta}|^{2n+2\alpha+2}} \\ &- \frac{(1-r^2)^{n+2\alpha}2(n+\alpha)(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta}))}{|1 - z\cdot\overline{\zeta}|^{2n+2\alpha+2}}\end{aligned} \tag{1.10}$$

To prove the right side inequality of (1.8), it suffices to prove

$$-2(n+2\alpha)r|1-z\cdot\overline{\zeta}|^2 - (1-r^2)(2n+2\alpha)(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})) \le (2n+2\alpha-2\alpha r)|1-z\cdot\overline{\zeta}|^2$$

which is equivalent to

$$-(1-r^2)2(n+\alpha)(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})) \le 2(n+\alpha)(1+r)|1 - z\cdot\overline{\zeta}|^2.$$

For $\alpha > -n$, the above inequality is equivalent to

$$-(1-r)((r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})) \le |1 - z\cdot\overline{\zeta}|^2$$

which is, after a simple simplification,

$$(1+r)\mathrm{Re}(\eta\cdot\overline{\zeta}) \le 1 + r|\eta\cdot\overline{\zeta}|^2.$$

The inequality is true by (1.6) in Lemma 1.12. To prove the left side inequality of (1.8), it suffices to show

$$-2(n+2\alpha)r|1-z\cdot\overline{\zeta}|^2 - (1-r^2)(2n+2\alpha)(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})) \ge -(2n+2\alpha+2\alpha r)|1-z\cdot\overline{\zeta}|^2$$

which is equivalent to

$$-(1-r^2)2(n+\alpha)(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})) \ge -2(n+\alpha)(1-r)|1 - z\cdot\overline{\zeta}|^2.$$

For $\alpha > -n$, the above inequality becomes

$$(1+r)((r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})) \le |1 - z\cdot\overline{\zeta}|^2$$

which is, after a simple simplification,

$$(-1+r)\mathrm{Re}(\eta\cdot\overline{\zeta}) \le 1 - r|\eta\cdot\overline{\zeta}|^2.$$

The inequality is true by (1.7) in Lemma 1.12. The proof of (1.9) is parallel to that of (1.8), using the same inequalities in Lemma 1.12. This completes the proof of Lemma 1.13. □

Lemma 1.14. *Let $u(z)$ be a positive invariant harmonic function in B^n defined by a positive Borel measure on S^{n-1} with the Poisson-Szegö kernel, $|z| = r$.*
If $\alpha > -n$,

$$-\frac{(2n+2\alpha+2\alpha r)}{1-r^2}u(z) \le \frac{\partial u(z)}{\partial r} \le \frac{(2n+2\alpha-2\alpha r)}{1-r^2}u(z). \tag{1.11}$$

If $\alpha < -n$,

$$\frac{(2n+2\alpha-2\alpha r)}{1-r^2}u(z) \le \frac{\partial u(z)}{\partial r} \le -\frac{(2n+2\alpha+2\alpha r)}{1-r^2}u(z). \tag{1.12}$$

Proof. By the Poisson-Szegö integral representation of u in B^n,

$$u(z) = \int_{S^{n-1}} \frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} d\mu(\zeta)$$

for a positive Borel measure μ on S^{n-1}. By (1.8) in Lemma 1.13 and μ being a positive measure,

$$\begin{aligned}\int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}\right) d\mu(\zeta) &\le \int_{S^{n-1}} \frac{(n+2\alpha-2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} d\mu(\zeta) \\ &= \frac{(n+2\alpha-2\alpha r)}{1-r^2}\int_{S^{n-1}} \frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} d\mu(\zeta) \\ &= \frac{(n+2\alpha-2\alpha r)}{1-r^2}u(z)\end{aligned}$$

when $\alpha > -n$. It follows that

$$\frac{\partial u(z)}{\partial r} = \int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}\right) d\mu(\zeta) \le \frac{(n+2\alpha-2\alpha r)}{1-r^2}u(z).$$

The left side inequality in (1.11) is proved similarly. For the equality case, consider $u_w(z) = P_\alpha(z,w) = \dfrac{(1-|z|^2)^{n+2\alpha}}{|z-w|^{2n+2\alpha}}$. It is known that $u_w(z)$ is invariant harmonic in $\mathbb{C}^n \setminus \{w\}$ for $w \in S^{n-1}$. A simple calculation shows that the equalities in (1.11) hold for $u_w(z)$ when $z = |z|w$ and $z = -|z|w$ respectively. The proof of (1.12) is parallel to that of (1.11), using (1.9) instead of (1.8) in Lemma 2.8. This completes the proof of Lemma 1.14. □

Now we prove Theorem 1.8.

Proof. Consider $\varphi(r) = \dfrac{(1-r)^n}{(1+r)^{n+2\alpha}}$, $\psi(r) = \dfrac{(1+r)^n}{(1-r)^{n+2\alpha}}$ for $0 \le r < 1$.

$$\frac{\varphi'}{\varphi} = -\frac{2n+2\alpha-2\alpha r}{1-r^2},$$
$$\frac{\psi'}{\psi} = \frac{2n+2\alpha+2\alpha r}{1-r^2}.$$

Given $\omega \in S^{n-1}$, consider

$$I(r,\omega) = \varphi(r)u(r\omega),$$
$$J(r,\omega) = \psi(r)u(r\omega).$$

To show Theorem 1.8, it suffices to show that $I(r,\omega)$ is decreasing (increasing) and $J(r,\omega)$ is increasing (decreasing) in r when $\alpha > -n$ (when $\alpha < -n$). By (1.11) in Lemma 1.14, when $\alpha > -n$,

$$\begin{aligned}\frac{\partial}{\partial r}\log I(r,\omega) &= \frac{\varphi'}{\varphi} + \frac{u'_r}{u} \\ &= -\frac{2n+2\alpha-2\alpha r}{1-r^2} + \frac{u'_r}{u} \\ &\le -\frac{2n+2\alpha-2\alpha r}{1-r^2} + \frac{2n+2\alpha-2\alpha r}{1-r^2} \\ &= 0.\end{aligned}$$

Therefore $\log I(r,\omega)$ is decreasing in r, and so is $I(r,\omega)$. Similarly,

$$\begin{aligned}\frac{\partial}{\partial r}\log J(r,\omega) &= \frac{\psi'}{\psi} + \frac{u'_r}{u} \\ &= \frac{2n+2\alpha+2\alpha r}{1-r^2} + \frac{u'_r}{u} \\ &\ge \frac{2n+2\alpha+2\alpha r}{1-r^2} - \frac{2n+2\alpha+2\alpha r}{1-r^2} \\ &= 0.\end{aligned}$$

Hence, $J(r,\omega)$ is increasing in r. For $\alpha > -n$ and $\zeta, w \in S^{n-1}$,

$$\begin{aligned}\lim_{r\to 1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}P_\alpha(r\zeta,w) &= \lim_{r\to 1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}\frac{(1-|r|^2)^{n+2\alpha}}{|1-r\zeta\cdot\overline{w}|^{2n+2\alpha}} \\ &= \lim_{r\to 1}\frac{(1-r)^{2n+2\alpha}}{|1-r\zeta\cdot\overline{w}|^{2n+2\alpha}} \searrow \delta(\zeta,w)\end{aligned}$$

by applying the monotonicity results in Theorem 1.8 to $u = P_\alpha$. By Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{r\to 1}(1-r)^n u(r\zeta) &= \lim_{r\to 1}(1-r)^n\int_{S^{n-1}} P_\alpha(r\zeta,\xi)d\mu(\xi) \\ &= \lim_{r\to 1}(1+r)^{n+2\alpha}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}P_\lambda(r\zeta,\xi)d\mu(\xi) \\ &= 2^{n+2\alpha}\mu(\{\zeta\}).\end{aligned}$$

Similarly, $\dfrac{(1+r)^n}{(1-r)^{n+2\alpha}} P_\alpha(r\zeta, w) = \dfrac{(1+r)^{2n+2\alpha}}{|1 - r\zeta \cdot \overline{w}|^{2n+2\alpha}}$ is increasing in r for $\alpha > -n$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}
\lim_{r\to 1} \frac{u(r\zeta)}{(1-r)^{n+2\alpha}} &= \lim_{r\to 1} \frac{1}{(1-r)^{n+2\alpha}} \int_{S^{n-1}} P_\alpha(r\zeta, \xi) d\mu(\xi) \\
&= \lim_{r\to 1} \frac{1}{(1+r)^n} \int_{S^{n-1}} \lim_{r\to 1} \frac{(1+r)^n}{(1-r)^{n+2\alpha}} P_\alpha(r\zeta, \xi) d\mu(\xi) \\
&= \frac{1}{2^n} \int_{S^{n-1}} \lim_{r\to 1} \frac{(1+r)^{2n+2\alpha}}{|1 - r\zeta \cdot \bar{\xi}|^{2n+2\alpha}} d\mu(\xi) \\
&= \int_{S^{n-1}} \frac{2^{n+2\alpha}}{|\zeta - \xi|^{2n+2\alpha}} d\mu(\xi).
\end{aligned}$$

For $\alpha < -n$, the monotonicity of $I(r, \omega)$ and $J(r, \omega)$ is proved similarly by applying (1.12) in Lemma 1.14.

$$\lim_{r\to 1} \frac{(1-r)^n}{(1+r)^{n+2\alpha}} P_\alpha(r\zeta, w) = \lim_{r\to 1} \frac{|1 - r\zeta \cdot \overline{w}|^{-(2n+2\alpha)}}{(1-r)^{-(2n+2\alpha)}} \nearrow \begin{cases} 1, & \zeta = w \\ \infty, & \zeta \neq w \end{cases}$$

when $\alpha < -n$. Therefore,

$$\begin{aligned}
\lim_{r\to 1} (1-r)^n u(r\zeta) &= \lim_{r\to 1} (1+r)^{n+2\alpha} \int_{S^{n-1}} \lim_{r\to 1} \frac{(1-r)^n}{(1+r)^{n+2\alpha}} P_\alpha(r\zeta, \xi) d\mu(\xi) \\
&= \begin{cases} 2^{n+2\alpha} \mu(\{\zeta\}), & if\ \mu(\{\zeta\}^c) = 0; \\ \infty, & if\ \mu(\{\zeta\}^c) > 0. \end{cases}
\end{aligned}$$

Similarly, $\dfrac{(1+r)^n}{(1-r)^{n+2\alpha}} P_\alpha(r\zeta, w) = \dfrac{(1+r)^{2n+2\alpha}}{|1 - r\zeta \cdot \overline{w}|^{2n+2\alpha}}$ is decreasing in r for $\alpha < -n$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}
\lim_{r\to 1} \frac{u(r\zeta)}{(1-r)^{n+2\alpha}} &= \lim_{r\to 1} \frac{1}{(1-r)^{n+2\alpha}} \int_{S^{n-1}} P_\alpha(r\zeta, \xi) d\mu(\xi) \\
&= \lim_{r\to 1} \frac{1}{(1+r)^n} \int_{S_{n-1}} \lim_{r\to 1} \frac{(1+r)^n}{(1-r)^{n+2\alpha}} P_\alpha(r\zeta, \xi) d\mu(\xi) \\
&= \frac{1}{2^n} \int_{S^{n-1}} \lim_{r\to 1} \frac{(1+r)^{2n+2\alpha}}{|1 - r\zeta \cdot \bar{\xi}|^{2n+2\alpha}} d\mu(\xi) \\
&= \int_{S^{n-1}} \frac{2^{n+2\alpha}}{|\zeta - \xi|^{2n+2\alpha}} d\mu(\xi)
\end{aligned}$$

This completes the proof of Theorem 1.8. □

Remark. Notice that the monotonicity of the auxiliary functions φ and ψ in the proofs of Theorem 1.1 and Theorem 1.8 may vary depending on the values of the parameter λ (or α) and the dimension n. When $\lambda > -\frac{n}{2}$ (or $\alpha > -n$), we have $\varphi' < 0$ and $\psi' > 0$, i.e. φ increases and ψ decreases in r for $0 < r < 1$. For $\lambda < -\frac{n}{2}$

(or $\alpha < -n$), the monotonicity does not necessarily hold. For example, in the real case in Theorem 1.1, for $\lambda < -\frac{n}{2}$,

$$\varphi(r) = \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\ , \qquad \varphi'(r) \begin{cases} > 0, & r \in \left(0, \dfrac{-2\lambda - n}{-2\lambda + (n-2)}\right) \\ < 0, & r \in \left(\dfrac{-2\lambda - n}{-2\lambda + (n-2)}, 1\right) \end{cases}$$

i.e. the monotonicity may change for certain combinations of n and λ. However, the monotonicity of $\varphi(r)u(r\zeta)$ and $\psi(r)u(r\zeta)$ holds.

1.4. **Proofs of Theorem 1.2 and Theorem 1.9.** The proofs for the two theorems are based on the following lemma.

Lemma 1.15. *Let $f(r)$ be a positive function on $r \in [0,1)$. If for $a, b \in \mathbb{R}$,*

$$-\frac{a+br}{1-r^2} f(r) \le f'(r) \le \frac{a-br}{1-r^2} f(r), \tag{1.13}$$

then for $0 \le r' \le r < 1$,

$$\left(\frac{1+r}{1+r'}\right)^{-a} \left(\frac{1-r^2}{1-r'^2}\right)^{\frac{b+a}{2}} f(r') \le f(r) \le \left(\frac{1+r}{1+r'}\right)^{a} \left(\frac{1-r^2}{1-r'^2}\right)^{\frac{b-a}{2}} f(r'). \tag{1.14}$$

Proof.

$$\int \frac{a-br}{1-r^2} dr = a\ln(1+r) + \frac{1}{2}(b-a)\ln(1-r^2) + C$$

Thus for $0 \le r' \le r'' < 1$, by (1.13),

$$\ln f(r'') - \ln f(r') = \int_{r'}^{r''} \frac{f'(r)}{f(r)} dr \le \int_{r'}^{r''} \frac{a-br}{1-r^2} dr \le \ln \left(\frac{1+r''}{1+r'}\right)^{a} \left(\frac{1-r''^2}{1-r'^2}\right)^{\frac{b-a}{2}}$$

i.e. the right side inequality in (1.14) holds. Similarly, by the left side of (1.13),

$$\ln f(r'') - \ln f(r') \ge \int_{r'}^{r''} -\frac{a+br}{1-r^2} dr \ge \ln \left(\frac{1+r''}{1+r'}\right)^{-a} \left(\frac{1-r''^2}{1-r'^2}\right)^{\frac{b+a}{2}}.$$

i.e. the left side inequality in (1.14) holds. $\square$

Now we prove Theorem 1.2.

Proof. If $\lambda > -\frac{n}{2}$, $u(r\zeta)$ satisfies (1.4) in Lemma 2.2. Therefore (1.13) holds with $f(r) = u(r\zeta)$, $a = n + 2\lambda$, $b = -n + 2\lambda + 2$. Let $0 \le r' \le r < 1$. (1.14) in Lemma 1.15 implies

$$\left(\frac{1+r}{1+r'}\right)^{-n-2\lambda} \left(\frac{1-r^2}{1-r'^2}\right)^{2\lambda+1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{n+2\lambda} \left(\frac{1-r^2}{1-r'^2}\right)^{-n+1} u(r'\zeta).$$

If $\lambda < -\frac{n}{2}$, $u(r\zeta)$ satisfies (1.5) in Lemma 2.2, thus (1.13) holds with $f(r) = u(r\zeta),\ a = -n - 2\lambda,\ b = -n + 2\lambda + 2$. Applying (1.14),

$$\left(\frac{1+r}{1+r'}\right)^{n+2\lambda}\left(\frac{1-r^2}{1-r'^2}\right)^{-n+1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{-n-2\lambda}\left(\frac{1-r^2}{1-r'^2}\right)^{2\lambda+1} u(r'\zeta).$$

This completes the proof of Theorem 1.2. □

The proof of Theorem 1.9 is similar to that of Theorem 1.2.

Proof. If $\alpha > -n$, $u(r\zeta)$ satisfies (1.11) in Lemma 1.14. Therefore (1.13) holds with $f(r) = u(r\zeta),\ a = 2n + 2\alpha,\ b = 2\alpha$. Let $0 \le r' \le r < 1$. (1.14) in Lemma 1.15 implies

$$\left(\frac{1+r}{1+r'}\right)^{-2n-2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{n+2\alpha} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2n+2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{-n} u(r'\zeta).$$

If $\alpha < -n$, $u(r\zeta)$ satisfies (1.12) in Lemma 2.9, thus (1.13) holds with $f(r) = u(r\zeta),\ a = -2n - 2\alpha,\ b = 2\alpha$. From (1.14),

$$\left(\frac{1+r}{1+r'}\right)^{2n+2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{-n} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{-2n-2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{n+2\alpha} u(r'\zeta).$$

This completes the proof of Theorem 1.9. □

Most results in this paper are on the function values at two points in B^n on the same ray. Similar results can be obtained for any two points in B^n (ref. [15] and Section 2).

2. On Harnack inequality for positive invariant harmonic functions

A refined estimate of Harnack inequality is proved for positive invariant harmonic functions.

2.1. **Statement of results.** Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\},\ n \geq 2$ be the unit ball in $\mathbb{R}^n, S^{n-1} = \partial B^n$. We prove a refined estimate of Harnack inequality for positive invariant harmonic functions defined by positive Borel measures on the sphere with respect to the Poisson kernel P_λ (defined below).

First we consider positive harmonic functions, and the more general case follows.

Theorem 2.1. *Let u be a positive harmonic function in B^n. $\xi_1, \xi_2 \in S^{n-1},\ 0 \leq r_1 \leq r_2 < 1$. Then*

$$f(-r_1, -r_2)\exp\{-g(r_1)\} \leq \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \leq f(r_1, r_2)\exp\{g(r_1)\}. \tag{2.1}$$

where

$$\begin{aligned} f(r_1, r_2) &= \left(\frac{1+r_2}{1+r_1}\right)\left(\frac{1-r_1}{1-r_2}\right)^{n-1} \\ g(r_1) = g(r_1, \xi_1, \xi_2) &= \frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr_1}{(1-r_1)^2} \end{aligned}$$

Remark. When $r_2 = r = |x|, r_1 = 0$, (2.1) becomes

$$\frac{1-r}{(1+r)^{n-1}} \leq \frac{u(x)}{u(0)} \leq \frac{1+r}{(1-r)^{n-1}}$$

— the classical Harnack inequality in B^n.

Denote the differential operator

$$\Delta_\lambda = (1-|x|^2)\left\{\frac{1-|x|^2}{4}\sum_j \frac{\partial^2}{\partial x_j^2} + \lambda \sum_j x_j \frac{\partial}{\partial x_j} + \lambda(\frac{n}{2} - 1 - \lambda)\right\},\ \lambda \in \mathbb{R}.$$

Invariant harmonic functions are solutions of $\Delta_\lambda u = 0$ and are of certain invariant property with respect to Möbius transformations. Let μ be a positive Borel measure on S^{n-1} and P_λ be the Poisson kernel

$$P_\lambda = \frac{(1-|x|^2)^{1+2\lambda}}{|x-\eta|^{n+2\lambda}}\ , \quad \lambda \in \mathbb{R}.$$

It is known that

$$u(x) = \int_{S^{n-1}} P_\lambda(x, \eta)d\mu(\eta)$$

is an invariant harmonic function in B^n ([1], p. 119).

Theorem 2.2. *Let u be a positive invariant harmonic function in B^n defined by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. Let $\xi_1, \xi_2 \in S^{n-1}$ and $0 \le r_1 \le r_2 < 1$.*

If $\lambda > -\frac{n}{2}$,

$$f_\lambda(-r_1,-r_2)\exp\{-g_\lambda(r_1)\} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le f_\lambda(r_1,r_2)\exp\{g_\lambda(r_1)\}. \tag{2.2}$$

If $\lambda < -\frac{n}{2}$,

$$f_\lambda(r_1,r_2)\exp\{-g_\lambda(r_1)\} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le f_\lambda(-r_1,-r_2)\exp\{g_\lambda(r_1)\} \tag{2.3}$$

where

$$\begin{aligned} f_\lambda(r_1,r_2) &= \left(\frac{1+r_2}{1+r_1}\right)^{2\lambda+1}\left(\frac{1-r_1}{1-r_2}\right)^{n-1} \\ g_\lambda(r_1) = g_\lambda(r_1,\xi_1,\xi_2) &= \frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r_1}{(1-r_1)^2} \end{aligned}$$

Case $\lambda = \frac{n}{2}-1$ corresponds to the Laplace-Beltrami operator $\Delta_{\frac{n}{2}-1}$ and the Poincaré metric. It is known ([2]) that, for positive u, $\Delta_{\frac{n}{2}-1}u = 0$, there exists a positive Borel measure μ on S^{n-1} such that

$$u(x) = \int_{S^{n-1}} P_{\frac{n}{2}-1}(x,\eta)d\mu(\eta).$$

In this case, Theorem 2.2 has the following form.

Corollary 2.3. *Let u be a positive solution of $\Delta_{\frac{n}{2}-1}u = 0$ in B^n. Let $\xi_1, \xi_2 \in S^{n-1}$ and $0 \le r_1 \le r_2 < 1$. Then*

$$\frac{1}{C} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le C,$$

where

$$C = C(r_1,r_2,\xi_1,\xi_2) = \left(\frac{1+r_2}{1+r_1}\cdot\frac{1-r_1}{1-r_2}\right)^{n-1}\exp\left\{\pi|\xi_2-\xi_1|\frac{(n-1)r_1}{(1-r_1)^2}\right\}.$$

The proofs will need an earlier result in Part 1 (also [14]) on the monotonicity of positive invariant harmonic functions. Here we state the result as a proposition.

Proposition 2.4. *(Theorem 1.2 in Par 1 and in* [14]*)*
Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. Let $\zeta \in S^{n-1}$ and $0 \le r' \le r < 1$.

If $\lambda > -\frac{n}{2}$,

$$\left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1}u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1}u(r'\zeta).$$

If $\lambda < -\frac{n}{2}$,

$$\left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1} u(r'\zeta) \leq u(r\zeta) \leq \left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1} u(r'\zeta).$$

2.2. **Proof of Theorem 2.1.** We need five lemmas before proving Theorem 2.1.

Let $x \cdot y = \sum_{k=1}^{n} x_k y_k$ denote the inner product in $\mathbb{R}^n$.

Lemma 2.5. *Let* $\eta, \xi_1, \xi_2 \in S^{n-1}$. *Let* $\varphi(t),\ t \in [0,1]$ *be the shortest arc on the great circle connecting* ξ_1 *and* ξ_2. *Then*

$$\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n} = \frac{nr\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2}} \tag{2.4}$$

Proof.

$$\frac{d}{dt}|r\varphi(t)-\eta|^2 = \frac{d}{dt}\left(r^2+1-2r\varphi(t)\cdot\eta\right) = -2r\phi'(t)\cdot\eta$$

$$\begin{aligned}
\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n} &= \frac{d}{dt}\left(|r\varphi(t)-\eta|^2\right)^{-n/2} \\
&= -\frac{n}{2}\left(|r\varphi(t)-\eta|^2\right)^{-\frac{n}{2}-1}\frac{d}{dt}|r\varphi(t)-\eta|^2 \\
&= \frac{nr\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2}}
\end{aligned}$$

□

Lemma 2.6. *Let* u *be a positive harmonic function in* B^n. $\xi_1, \xi_2 \in S^{n-1}$. *Let* $\varphi(t),\ t \in [0,1]$ *be the shortest arc on the great circle connecting* ξ_1 *and* ξ_2. *Then*

$$\left|\frac{d}{dt}u(r\varphi(t))\right| \leq \frac{nr|\varphi'(t)|}{(1-r)^2}u(r\varphi(t)), \qquad r \in [0,1].$$

Proof. By (2.4),

$$\begin{aligned}
\int_{S^{n-1}}\left|\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n}\right|d\mu(\eta) &= \int_{S^{n-1}}\left|\frac{nr\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2}}\right|d\mu(\eta) \\
&\leq nr\int_{S^{n-1}}\frac{|\varphi'(t)|\ |\eta|}{|r\varphi(t)-\eta|^n(1-r)^2}d\mu(\eta) \\
&= \frac{nr|\varphi'(t)|}{(1-r)^2(1-r^2)}\int_{S^{n-1}}\frac{1-r^2}{|r\varphi(t)-\eta|^n}d\mu(\eta) \\
&= \frac{nr|\varphi'(t)|}{(1-r)^2(1-r^2)}u(r\varphi(t))
\end{aligned}$$

where we applied the inequality

$$|r\varphi(t)-\eta| = |1 - r\phi(t)\cdot\eta| \geq 1-r.$$

Therefore by the Lebesgue's Dominant Convergence Theorem,

$$\begin{aligned}\left|\frac{d}{dt}u(r\varphi(t))\right| &= \left|\frac{d}{dt}\int_{S^{n-1}}\frac{1-|r\varphi(t)|^2}{|r\varphi(t)-\eta|^n}d\mu(\eta)\right| \\ &= (1-r^2)\left|\int_{S^{n-1}}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n}d\mu(\eta)\right| \\ &\leq \frac{nr|\varphi'(t)|}{(1-r)^2}u(r\varphi(t)).\end{aligned}$$

□

Lemma 2.7. *Let $\xi_1, \xi_2 \in S^{n-1}$. Then there exists a Möbius transformation T in $\mathbb{R}^n$ such that $T(S^{n-1}) = S^{n-1}$ and for all $r \in [0,1]$,*

$$T(r\xi_i) = (0,\cdots,0,r\cos\theta_i, r\sin\theta_i),\ i=1,2,\ |\theta_2-\theta_1|\leq\pi$$

and

$$|\det T'(x)| = 1, \quad |T(r\xi_2)-T(r\xi_1)| = |r\xi_2 - r\xi_1|, \tag{2.5}$$

where $T'(x)$ denotes the Jacobian matrix of the Möbius transformation T.

Proof. The transformations involves a rotation in $\mathbb{R}^n$ with respect to the origin such that ξ_1, ξ_2 are in the $\mathbb{R}^2$ plane of the last two coordinates. If $|\theta_2-\theta_1|\leq\pi$ is not yet satisfied, it can be achieved by a reflection with respect to the origin. Both rotation and reflection preserve the Euclidean norm and distance, and the absolute value of the determinant of the Jacobian matrix $|\det T'(x)| = 1$. □

Lemma 2.8. *Let*

$$\xi_i = (0,\cdots,0,r\cos\theta_i, r\sin\theta_i) \in S^{n-1},\ i=1,2,\ |\theta_2-\theta_1|\leq\pi$$

and

$$\varphi(t) = (0,\cdots,0,r\cos\theta_t, r\sin\theta_t),\ \theta_t = t\theta_2 + (1-t)\theta_1,\ t\in[0,1].$$

Then

$$|\varphi'(t)| \leq \frac{\pi}{2}|\xi_2-\xi_1|. \tag{2.6}$$

Proof. It suffices to prove for $n=2$. For computation convenience, we use the complex plane notations in $\mathbb{R}^2$. Denote $\xi_i = e^{1\theta_i}$.

$$\begin{aligned}\xi_2-\xi_1 &= e^{i\theta_2}-e^{i\theta_1} \\ &= \exp\left(i\frac{\theta_2+\theta_1}{2}\right)\left(\exp\left(i\frac{\theta_2-\theta_1}{2}\right)-\exp\left(-i\frac{\theta_2-\theta_1}{2}\right)\right) \\ &= \exp\left(i\frac{\theta_2+\theta_1}{2}\right)(2i)\sin\frac{\theta_2-\theta_1}{2}\end{aligned}$$

Notice that

$$\left|\sin\frac{x}{2}\right| \geq \left|\frac{x}{\pi}\right| \quad for \quad |x| \leq \pi,$$

so $|\theta_2 - \theta_1| \leq \pi$ implies

$$\left|e^{i\theta_2} - e^{i\theta_1}\right| = 2\left|\sin\left(\frac{\theta_2 - \theta_1}{2}\right)\right| \geq \frac{2}{\pi}|\theta_2 - \theta_2|.$$

Furthermore,

$$\varphi'(t) = \frac{d}{dt}\left(e^{it\theta_2 + i(1-t)\theta_1}\right) = \varphi(t)i(\theta_2 - \theta_1).$$

Thus

$$|\varphi'(t)| = |\theta_2 - \theta_1| \leq \frac{\pi}{2}\left|e^{i\theta_2} - e^{i\theta_1}\right| = \frac{\pi}{2}|\xi_2 - \xi_1|.$$

□

Lemma 2.9. *Let u be a positive invariant harmonic function. $\xi_1, \xi_2 \in S^{n-1}$. Then for $r \in [0,1)$,*

$$\exp\left\{-\frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2}\right\} \leq \frac{u(r\xi_2)}{u(r\xi_1)} \leq \exp\left\{\frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2}\right\}.$$

Proof. Let T be the Möbius transformation in $\mathbb{R}^n$ such that for $r \in [0,1]$,

$$r\zeta_i = T(r\xi_i) = (0, \cdots, 0, r\cos\theta_i, r\sin\theta_i),\ i = 1, 2,\ |\theta_2 - \theta_1| \leq \pi.$$

By (2.5) in Lemma 2.3 and [11],

$$U(x) = u(T^{-1}(x))$$

is still a positive harmonic function in B^n with respect to the measure $\mu(T^{-1}(x))$, $|\det T'(x)| = 1$. Furthermore,

$$U(r\zeta_i) = u(T^{-1}(r\zeta_i)) = u(r\xi_i),\ i = 1, 2$$

and

$$|\xi_2 - \xi_1| = |T(\xi_2) - T(\xi_1)| = |\zeta_2 - \zeta_1| = \left|e^{i\theta_2} - e^{i\theta_1}\right|.$$

Let

$$\varphi(t) = (0, \cdots, 0, r\cos\theta_t, r\sin\theta_t),\ \theta_t = t\theta_2 + (1-t)\theta_1,\ t \in [0,1]$$

be the shortest arc on the great circle connecting ζ_1 and ζ_2, $\varphi(0) = \zeta_1, \varphi(1) = \zeta_2$. By (2.6) in Lemma 2.8,

$$\begin{aligned}\left|\int_0^1 \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}dt\right| &\leq \int_0^1 \left|\frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}\right| dt \\ &\leq \frac{nr}{(1-r)^2}\int_0^1 |\varphi'(t)|dt \\ &\leq \frac{\pi}{2}|\zeta_2 - \zeta_1|\frac{nr}{(1-r)^2}\end{aligned}$$

Since

$$\ln\frac{u(r\xi_2)}{u(r\xi_1)} = \ln\frac{U(r\zeta_2)}{U(r\zeta_1)} = \ln\frac{U(r\varphi(1))}{U(r\varphi(0))} = \int_0^1 \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}dt,$$

we have

$$\left|\ln\frac{u(r\xi_2)}{u(r\xi_1)}\right| \le \frac{\pi}{2}|\zeta_2-\zeta_1|\frac{nr}{(1-r)^2} = \frac{\pi}{2}|\xi_2-\xi_1|\frac{nr}{(1-r)^2}.$$

Therefore

$$-\frac{\pi}{2}|\xi_2-\xi_1|\frac{nr}{(1-r)^2} \le \ln\frac{u(r\xi_2)}{u(r\xi_1)} \le \frac{\pi}{2}|\xi_2-\xi_1|\frac{nr}{(1-r)^2}.$$

This completes the proof of Lemma 2.9. □

Now we prove Theorem 2.1.

Proof.

$$\frac{u(r_2\xi_2)}{u(r_1\xi_1)} = \frac{u(r_2\xi_2)}{u(r_1\xi_2)}\frac{u(r_1\xi_2)}{u(r_1\xi_1)}$$

Proposition 1.4 implies

$$\frac{1-r_2}{1-r_1}\left(\frac{1+r_1}{1+r_2}\right)^{n-1} \le \frac{u(r_2\xi_2)}{u(r_1\xi_2)} \le \frac{1+r_2}{1+r_1}\left(\frac{1-r_1}{1-r_2}\right)^{n-1}.$$

Combine the above with the results in Lemma 2.9. Theorem 2.1 follows. □

2.3. **Proof of Theorem 2.2.** We need the following three lemmas for the proof of Theorem 2.2.

Lemma 2.10. *Let $\eta, \xi_1, \xi_2 \in S^{n-1}$. Let $\varphi(t),\ t \in [0,1]$ be the shortest arc on the great circle connecting ξ_1 and ξ_2. Then for $\lambda \in \mathbb{R}$,*

$$\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}} = \frac{(n+2\lambda)r\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2\lambda+2}} \tag{2.7}$$

Proof. The proof is similar to that of Lemma 2.5.

$$\begin{aligned}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}} &= \frac{d}{dt}\left(|r\varphi(t)-\eta|^2\right)^{-\frac{n+2\lambda}{2}} \\ &= -\frac{n+2\lambda}{2}\left(|r\varphi(t)-\eta|^2\right)^{-\frac{n+2\lambda}{2}-1}\frac{d}{dt}|r\varphi(t)-\eta|^2 \\ &= \frac{(n+2\lambda)r\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2\lambda+2}}\end{aligned}$$

using the result from the proof of (2.4) in Lemma 2.5. □

Lemma 2.11. *Let u be a positive invariant harmonic function in B^n defined by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. $\xi_1, \xi_2 \in S^{n-1}$. Let $\varphi(t),\ t \in [0,1]$ be the shortest arc on the great circle connecting ξ_1 and ξ_2. Then for $\lambda \in \mathbb{R},\ \lambda \neq -\frac{n}{2}$,*

$$\left|\frac{d}{dt}u(r\varphi(t))\right| \leq \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2}u(r\varphi(t)), \qquad r \in [0,1]. \tag{2.8}$$

Proof. By (2.7) and $|r\varphi(t)-\eta| = |1-r\phi(t)\cdot\eta| \geq 1-r$,

$$\begin{aligned}
\int_{S^{n-1}}\left|\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}}\right|d\mu(\eta) &= \int_{S^{n-1}}\frac{|(n+2\lambda)r\varphi'(t)\cdot\eta|}{|r\varphi(t)-\eta|^{n+2\lambda+2}}d\mu(\eta)\\
&\leq |n+2\lambda|r\int_{S^{n-1}}\frac{|\varphi'(t)|\ |\eta|}{|r\varphi(t)-\eta|^{n+2\lambda}(1-r)^2}d\mu(\eta)\\
&= \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2(1-r^2)^{1+2\lambda}}\int_{S^{n-1}}\frac{(1-r^2)^{1+2\lambda}}{|r\varphi(t)-\eta|^{n+2\lambda}}d\mu(\eta)\\
&= \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2(1-r^2)^{1+2\lambda}}u(r\varphi(t)).
\end{aligned}$$

By the Lebesgue's Dominant Convergence Theorem,

$$\begin{aligned}
\left|\frac{d}{dt}u(r\varphi(t))\right| &= \left|\frac{d}{dt}\int_{S^{n-1}}\frac{(1-|r\varphi(t)|^2)^{1+2\lambda}}{|r\varphi(t)-\eta|^{n+2\lambda}}d\mu(\eta)\right|\\
&= (1-r^2)^{1+2\lambda}\left|\int_{S^{n-1}}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}}d\mu(\eta)\right|\\
&\leq \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2}u(r\varphi(t)).
\end{aligned}$$

□

Lemma 2.12. *Let u be a positive invariant harmonic function in B^n defined by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. $\xi_1, \xi_2 \in S^{n-1}$. Then for $r \in [0,1)$,*

$$\exp\left\{-\frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2}\right\} \leq \frac{u(r\xi_2)}{u(r\xi_1)} \leq \exp\left\{\frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2}\right\} \tag{2.9}$$

Proof. Let T be the Möbius transformation in $\mathbb{R}^n$ such that

$$r\zeta_i = T(r\xi_i) = (0,\cdots,0,r\cos\theta_i, r\sin\theta_i),\ i=1,2,\ |\theta_2-\theta_1| \leq \pi.$$

By (2.5) in Lemma 2.7 and [11],

$$U(x) = u(T^{-1}(x))$$

is also a positive invariant harmonic function in B^n with respect to the measure $\mu(T^{-1}(x))$, $|\det T'(x)| = 1$.

$$U(r\zeta_i) = u(T^{-1}(r\zeta_i)) = u(r\xi_i),\ i=1,2$$

and

$$|\xi_2 - \xi_1| = |T(\xi_2) - T(\xi_1)| = |\zeta_2 - \zeta_1| = |e^{i\theta_2} - e^{i\theta_1}|.$$

Let

$$\varphi(t) = (0, \cdots, 0, r\cos\theta_t, r\sin\theta_t),\ \ \theta_t = t\theta_2 + (1-t)\theta_1,\ \ t \in [0,1].$$

Then $\varphi(0) = \zeta_1, \varphi(1) = \zeta_2$. By (2.8) in Lemma 2.11,

$$\begin{aligned} \left| \int_0^1 \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))} dt \right| &\leq \int_0^1 \left| \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))} \right| dt \\ &\leq \frac{|n+2\lambda| r}{(1-r)^2} \int_0^1 |\varphi'(t)| dt \\ &\leq \frac{\pi}{2} |\zeta_2 - \zeta_1| \frac{|n+2\lambda| r}{(1-r)^2}. \end{aligned}$$

Since

$$\ln \frac{u(r\xi_2)}{u(r\xi_1)} = \ln \frac{U(r\zeta_2)}{U(r\zeta_1)} = \ln \frac{U(r\varphi(1))}{U(r\varphi(0))} = \int_0^1 \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))} dt,$$

we have

$$\left| \ln \frac{u(r\xi_2)}{u(r\xi_1)} \right| \leq \frac{\pi}{2} |\zeta_2 - \zeta_1| \frac{|n+2\lambda| r}{(1-r)^2} = \frac{\pi}{2} |\xi_2 - \xi_1| \frac{|n+2\lambda| r}{(1-r)^2}.$$

Therefore

$$-\frac{\pi}{2} |\xi_2 - \xi_1| \frac{|n+2\lambda| r}{(1-r)^2} \leq \ln \frac{u(r\xi_2)}{u(r\xi_1)} \leq \frac{\pi}{2} |\xi_2 - \xi_1| \frac{|n+2\lambda| r}{(1-r)^2}.$$

This completes the proof of Lemma 2.12. □

The proof Theorem 2.2 is similar to that of Theorem 2.1.

Proof.

$$\frac{u(r_2\xi_2)}{u(r_1\xi_1)} = \frac{u(r_2\xi_2)}{u(r_1\xi_2)} \frac{u(r_1\xi_2)}{u(r_1\xi_1)}$$

Apply (2.9) in Lemma 2.12 and Proposition 2.4. Theorem 2.2 follows. □

3. On higher order angular derivatives — an application of Faá di Bruno's formula

3.1. **Statements of results.** We study the singular behavior of kth angular derivatives of analytic functions in the unit disk in the complex plane $\mathbb{C}$ and positive harmonic functions in the unit ball in $\mathbb{R}^n$. Faá di Bruno's formula plays an important role in our proofs.

Let $\mathbb{D} = \{z : \ |z| < 1\} \subset \mathbb{C}$ and $\mathbb{T} = \partial\mathbb{D}$. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and $\zeta \in \mathbb{T}$. ζ is a fixed point of φ if $\lim_{r\to 1} \varphi(r\zeta) = \zeta$. The angular derivative at ζ is defined as $\varphi'(\zeta) = \lim_{r\to 1} \varphi'(r\zeta)$. It is a consequence of the Julia Lemma [16] that the angular derivative at the fixed point exists and that $\varphi'(\zeta) \in (0, \infty]$. When the angular derivative of a fixed point is finite, what could be the limiting behavior of the higher order angular derivatives? We describe an asymptotic property of the higher order derivatives of the fixed point in the following theorem.

Theorem 3.1. *Let $\varphi(z) : \mathbb{D} \to \mathbb{D}$ be analytic. Let $\zeta \in \mathbb{T}$ be a fixed point of φ with angular derivative $\varphi'(\zeta) < \infty$. Then $\forall \ell \geq 2$, the ℓ-th angular derivative*

$$\varphi^{(\ell)}(r\zeta) = o\left(\frac{1}{(1-r)^{\ell-1}}\right) \qquad as \quad r \to 1. \tag{3.1}$$

The above is equivalent to $\lim_{r\to 1}(1-r)^{\ell-1}\varphi^{(\ell)}(r\zeta) = 0$ by the definition of the little o notation.

The order $\ell - 1$ in Theorem 3.1 is sharp in the sense illustrated in the following proposition and its proof.

Proposition 3.2. *For any $\varepsilon \in (0, 1)$ and $\zeta \in \mathbb{T}$, there exists an analytic function $\psi(z) : \mathbb{D} \to \mathbb{D}$ such that ζ is a fixed point of ψ, $\psi'(\zeta) < \infty$, and*

$$\lim_{r\to 1}(1-r)^{\ell-1-\varepsilon}|\psi^{(\ell)}(r\zeta)| > 0, \qquad \forall \ell \geq 2. \tag{3.2}$$

Furthermore, for any integer $m \geq 1$, there exists an analytic function $\psi(z) : \mathbb{D} \to \mathbb{D}$ such that ζ is a fixed point of ψ,

$$\begin{aligned}
&\psi^{(m-j)}(\zeta) < \infty, \qquad \forall j \geq 0,\\
&\psi^{(m+k)}(\zeta) = \infty, \qquad \forall k \geq 1,\\
&\lim_{r\to 1}(1-r)^{m+k-1}\psi^{(n+k)}(r\zeta) = 0,\\
&\lim_{r\to 1}(1-r)^{m+k-1-\varepsilon}|\psi^{(m+k)}(r\zeta)| > 0.
\end{aligned} \tag{3.3}$$

Results analogous to Theorem 3.1 can be obtained for positive harmonic functions, as stated in the following theorem.

Theorem 3.3. *Let u be a positive harmonic function in the unit ball $B^n \subset \mathbb{R}^n, n \geq 2$. Let $\zeta \in S^{n-1} = \partial B^n$. Then for $k \geq 1$,*

$$\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}u(r\zeta)\right\} = 2\frac{(n+k-2)!}{(n-2)!}\lim_{r\to 1}\frac{(1-r)^{n-1}}{1+r}u(r\zeta). \tag{3.4}$$

Consequently,

$$\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}u(r\zeta)\right\} = 0 \tag{3.5}$$

except possibly on a countable set of points on the sphere.

From the proof of Theorem 3.3 we can see that the results can be extended to harmonic functions defined by complex measures. We may restate Theorem 3.3 as the following.

Theorem 3.3′. *Let u be a harmonic function in the unit ball $B^n, n \geq 2$ defined by a complex measure μ on S^{n-1} (with the Poisson kernel). Let $\zeta \in S^{n-1}$. Then for $k \geq 1$,*

$$\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}u(r\zeta)\right\} = 2\frac{(n+k-2)!}{(n-2)!}\lim_{r\to 1}\frac{(1-r)^{n-1}}{1+r}u(r\zeta).$$

3.2. **Proof of Theorem 3.1.** First we prove a lemma needed for the proof of Theorem 3.1.

Lemma 3.4. *Let $f(z)$ be analytic and $\mathrm{Re} f(z) > 0$ for $z \in \mathbb{D}$. Let $\zeta \in \mathbb{T}$. Then*

$$\lim_{r\to 1}(1-r)^{k+1}f^{(k)}(r\zeta) = \overline{\zeta}^k 2k!\ \lim_{r\to 1}\frac{1-r}{1+r}f(r\zeta).$$

Proof. The proof follows the steps similar to the proof of Theorem 1.3 in [4]. First consider the case $f(0) = 1$. Since $\mathrm{Re} f(z) > 0$, there exists a unique positive Borel measure μ such that (ref. [5])

$$f(z) = \int_{\mathbb{T}}\frac{1+\eta z}{1-\eta z}d\mu(\eta), \qquad \mu(\mathbb{T}) = 1.$$

Direct calculation yields

$$f^{(k)}(z) = 2k!\int_{\mathbb{T}}\frac{\eta^k}{(1-\eta z)^{k+1}}d\mu(\eta), \qquad k \geq 1.$$

Consider $z = r\zeta$. Since

$$\lim_{r\to 1}\frac{1-r}{1-r\zeta\eta} = \begin{cases}1, & \eta = \overline{\zeta};\\ 0, & \eta \neq \overline{\zeta};\end{cases}$$

we have

$$\int_{\mathbb{T}} \lim_{r\to 1} \frac{1-r}{1+r}\frac{1+r\zeta\eta}{1-r\zeta\eta} d\mu(\eta) = \mu(\{\overline{\zeta}\}), \qquad \int_{\mathbb{T}} \lim_{r\to 1} \frac{\eta^k(1-r)^{k+1}}{(1-r\zeta\eta)^{k+1}} d\mu(\eta) = \overline{\zeta}^k \mu(\{\overline{\zeta}\}), \quad k \geq 1.$$

By the Lebesgue's dominated convergence theorem,

$$\begin{aligned} \lim_{r\to 1}\left\{\frac{1-r}{1+r}f(r\zeta)\right\} &= \lim_{r\to 1}\left\{\frac{1-r}{1+r}\int_{\mathbb{T}}\frac{1+r\zeta\eta}{1-r\zeta\eta}d\mu(\eta)\right\} \\ &= \int_{\mathbb{T}}\lim_{r\to 1}\left\{\frac{1-r}{1+r}\frac{1+r\zeta\eta}{1-r\zeta\eta}\right\}d\mu(\eta) = \mu(\{\overline{\zeta}\}), \end{aligned}$$

and

$$\begin{aligned} \lim_{r\to 1}\left\{(1-r)^{k+1}f^{(k)}(r\zeta)\right\} &= \lim_{r\to 1}\left\{(1-r)^{k+1}2k!\int_{\mathbb{T}}\frac{\eta^k}{(1-\eta z)^{k+1}}d\mu(\eta)\right\} \\ &= 2k!\int_{\mathbb{T}}\lim_{r\to 1}\frac{\eta^k(1-r)^{k+1}}{(1-r\zeta\eta)^{k+1}}d\mu(\eta) = \overline{\zeta}^k\mu(\{\overline{\zeta}\}), \quad k\geq 1. \end{aligned}$$

Therefore,

$$\lim_{r\to 1}(1-r)^{k+1}f^{(k)}(r\zeta) = \overline{\zeta}^k 2k! \lim_{r\to 1}\frac{1-r}{1+r}f(r\zeta).$$

If $f(0)\neq 1$, consider

$$g(z) = \frac{f(z) - i\ \mathrm{Im}f(0)}{\mathrm{Re}f(0)},$$

we have

$$g^{(k)}(z) = \frac{f^{(k)}(z)}{\mathrm{Re}f(0)}, \quad g(0)=1, \quad \mathrm{Re}(g(z)) = \frac{\mathrm{Re}f(z)}{\mathrm{Re}f(0)} > 0 \quad for \quad z\in\mathbb{D}.$$

Thus

$$\begin{aligned} \lim_{r\to 1}(1-r)^{k+1}\frac{f^{(k)}(r\zeta)}{\mathrm{Re}f(0)} &= \overline{\zeta}^k 2k! \lim_{r\to 1}\left\{\frac{1-r}{1+r}\frac{f(r\zeta)-i\ \mathrm{Im}f(0)}{\mathrm{Re}f(0)}\right\} \\ &= \overline{\zeta}^k 2k! \lim_{r\to 1}\left\{\frac{1-r}{1+r}\frac{f(r\zeta)}{\mathrm{Re}f(0)}\right\}. \end{aligned}$$

Consequently,

$$\lim_{r\to 1}(1-r)^{k+1}f^{(k)}(r\zeta) = \overline{\zeta}^k 2k! \lim_{r\to 1}\frac{1-r}{1+r}f(r\zeta).$$

□

The following is the proof of Theorem 3.1.

Proof. By considering the analytic function $\overline{\zeta}\varphi(\zeta z) : \mathbb{D}\to\mathbb{D}$, we only need to prove Theorem 3.1 for the case $\zeta = 1$ without loss of generality. Let

$$f(z) = \frac{1+\varphi(z)}{1-\varphi(z)}, \quad z\in\mathbb{D}.$$

Then

$$\text{Re}f(z) > 0 \qquad \textit{and} \qquad \varphi(z) = \frac{f(z)-1}{f(z)+1}, \qquad \forall z \in \mathbb{D}.$$

Furthermore,

$$\lim_{r\to 1}\frac{1-r}{1+r}f(r) = \lim_{r\to 1}\frac{1-r}{1+r}\frac{1+\varphi(r)}{1-\varphi(r)} = \lim_{r\to 1}\frac{1-r}{1-\varphi(r)}\frac{1+\varphi(r)}{1+r} = \frac{1}{\varphi'(1)}.$$

Subsequently,

$$\lim_{r\to 1}(1-r)(f(r)+1) = \frac{2}{\varphi'(1)}.$$

By Lemma 3.4,

$$\lim_{r\to 1}(1-r)^{k+1}\frac{f^{(k)}(r)}{k!} = \frac{1}{k!}\lim_{r\to 1}\frac{1-r}{1+r}f(r) = \frac{2}{\varphi'(1)} \qquad for \qquad k \geq 0.$$

Let $h(z) = \dfrac{z-1}{z+1}$, then $\varphi(z) = h(f(z))$. By Faà di Bruno's formula [7],

$$\varphi^{(\ell)}(r) = \frac{d^\ell}{dz^\ell}h(f(r)) \quad = \quad \sum \frac{\ell!}{m_1!\; m_2! \cdots m_\ell!}h^{(m_1+\cdots+m_\ell)}(f(r))\prod_j \left(\frac{f^{(j)}(r)}{j!}\right)^{m_j}$$

where the sum is over all ℓ-tuples $(m_1, m_2, \cdots, m_\ell)$ satisfying

$$1m_1 + 2m_2 + \cdots + \ell m_\ell = \ell.$$

Since

$$h^{(k)}(z) = \frac{2(-1)^{k+1}k!}{(z+1)^{k+1}}, \qquad k \geq 1,$$

we have

$$\begin{aligned}
\varphi^{(\ell)}(r) &= \sum \frac{\ell!}{m_1!\; m_2! \cdots m_\ell!}\frac{2(-1)^{(m_1+\cdots+m_\ell+1)}(m_1+\cdots+m_\ell)!}{(f(r)+1)^{m_1+\cdots+m_\ell+1}}\prod_j \left(\frac{f^{(j)}(r)}{j!}\right)^{m_j} \\
&= \ell!\sum \frac{(-1)^{(m_1+\cdots+m_\ell+1)}(m_1+\cdots+m_\ell)!}{m_1!\; m_2! \cdots m_\ell!}\frac{2}{f(r)+1}\prod_j \left(\frac{f^{(j)}(r)}{(f(r)+1)j!}\right)^{m_j}.
\end{aligned}$$

Notice that for each term of the sum, $1m_1 + 2m_2 + \cdots + \ell m_\ell = \ell$, therefore

$$\begin{aligned}
&\lim_{r\to 1}\left\{(1-r)^{\ell-1}\frac{2}{f(r)+1}\prod_j \left(\frac{f^{(j)}(r)}{(f(r)+1)j!}\right)^{m_j}\right\} \\
&= \lim_{r\to 1}\left\{\frac{2}{(1-r)(f(r)+1)}\prod_j \left(\frac{(1-r)^{j+1}f^{(j)}(r)/j!}{(1-r)(f(r)+1)}\right)^{m_j}\right\} \\
&= \frac{2}{2/\varphi'(1)}\prod_j \left(\frac{2/\varphi'(1)}{2/\varphi'(1)}\right)^{m_j} = \varphi'(1).
\end{aligned}$$

Consequently,

$$\lim_{r\to 1}\left\{(1-r)^{\ell-1}\varphi^{(\ell)}(r)\right\} \quad = \quad \varphi'(1)\ell!\sum \frac{(-1)^{(m_1+\cdots+m_\ell+1)}(m_1+\cdots+m_\ell)!}{m_1!\; m_2!\cdots m_\ell!}.$$

To see that the above sum is zero, consider the function

$$g(x)=x^{-1}, \quad g^{(k)}(x)=k!(-1)^k x^{-(k+1)}, \quad g^{(k)}(g(x))=-k!(-x)^{k+1}, \quad x\in(0,1].$$

Applying Faà di Bruno's formula to $x=g(g(x))$, we have

$$\begin{aligned}
\frac{d^\ell}{dx^\ell}(x) &= \frac{d^\ell}{dr^\ell}g(g(x)) \\
&= \sum \frac{\ell!}{m_1!\; m_2!\cdots m_\ell!} g^{(m_1+\cdots+m_\ell)}(g(r)) \prod_j \left(\frac{g^{(j)}(r)}{j!}\right)^{m_j} \\
&= \ell!\sum \frac{-(m_1+\cdots+m_\ell)!(-x)^{m_1+\cdots+m_\ell+1}}{m_1!\; m_2!\cdots m_\ell!} \prod_j \left(\frac{(-1)^j}{x^{j+1}}\right)^{m_j} \\
&= \ell!\sum \frac{-(m_1+\cdots+m_\ell)!(-1)^{m_1+\cdots+m_\ell+1}x}{m_1!\; m_2!\cdots m_\ell!} \; \frac{(-1)^\ell}{x^\ell} \\
&= \left(\frac{-1}{x}\right)^{\ell-1} \ell!\sum \frac{(-1)^{m_1+\cdots+m_\ell+1}(m_1+\cdots+m_\ell)!}{m_1!\; m_2!\cdots m_\ell!} \;\equiv\; 0, \qquad \forall x\in(0,1],\ \ell\geq 2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{r\to 1}\left\{(1-r)^{\ell-1}\varphi^{(\ell)}(r)\right\} &= \varphi'(1)\ell!\sum \frac{(-1)^{(m_1+\cdots+m_\ell+1)}(m_1+\cdots+m_\ell)!}{m_1!\; m_2!\cdots m_\ell!} \\
&= \varphi'(1)(-1)^{\ell-1}\left(\frac{d^\ell}{dx^\ell}(x)\bigg|_{x=1}\right) \equiv 0,
\end{aligned}$$

therefore

$$\varphi^{(\ell)}(r)=o\left(\frac{1}{(1-r)^{\ell-1}}\right) \quad as \quad r\to 1, \qquad \forall \ell\geq 2.$$

$\square$

3.3. **Proof of Proposition 3.2.** The following lemma is needed for the proof of Proposition 3.2.

Lemma 3.5. *Let* $\alpha \in (0,1)$,

$$\begin{aligned} f_1(x) &= \sum_{n=0}^{\infty} a_{2n+1} x^{2n+1}, \qquad a_{2n+1} = \binom{\alpha}{2n+1} = \frac{\alpha(\alpha-1)\cdots(\alpha-2n)}{(2n+1)!}, \\ f_2(x) &= \sum_{n=0}^{\infty} b_{2n} x^{2n}, \qquad b_{2n} = \binom{\alpha}{2n} = \frac{\alpha(\alpha-1)\cdots(\alpha-2n+1)}{(2n)!}, \\ h(x) &= \frac{f_1(x)}{f_2(x)} = \sum_{n=0}^{\infty} c_n x^n. \end{aligned}$$

Then

$$c_{2n} = 0, \qquad c_{2n+1} > 0, \qquad \forall n \geq 0.$$

Proof. By the symmetry of f_1 and f_2,

$$h(-x) = \frac{f_1(-x)}{f_2(-x)} = -\frac{f_1(x)}{f_2(x)} = -h(x) \quad \Longrightarrow \quad c_{2n} = 0, \ \forall n \geq 0.$$

Furthermore,

$$f_1(x) = h(x) f_2(x) \quad \Longrightarrow \quad a_{2n+1} = c_{2n+1} b_0 + c_{2n-1} b_2 + \cdots + c_1 b_{2n}, \quad \forall \ n \geq 0.$$

Since $b_0 = 1$, we have

$$c_{2n+1} = a_{2n+1} - (c_{2n-1} b_2 + \cdots + c_1 b_{2n}), \quad \forall n \geq 0.$$

For $n = 0, \ 1$,

$$\begin{aligned} c_1 &= a_1 = \alpha > 0, \\ c_3 &= a_3 - c_1 b_2 = \frac{\alpha(\alpha-1)(\alpha-2)}{3!} - \alpha \frac{\alpha(\alpha-1)}{2!} = \alpha(\alpha-1)\left(-\frac{1}{3}\right)(\alpha+1) > 0. \end{aligned}$$

Now assume $c_{2j-1} > 0$ for $j = 1, \cdots, k$ for some $k \geq 1$. Notice that for $\alpha \in (0,1)$,

$$a_{2k+1} > 0, \ \forall k \geq 0 \qquad \textit{and} \qquad b_{2k} < 0, \ \forall k \geq 1.$$

Therefore

$$c_{2k+1} = a_{2k+1} - c_{2k-1} b_2 - \cdots - c_1 b_{2k} > 0,$$

because every term is positive. By induction, $c_{2n+1} > 0, \ \forall n \geq 0.$ □

The following is the proof of Proposition 3.2.

Proof. We prove (3.2) in Proposition 3.2 by constructing a function ψ such that ψ (or its rotation) satisfies (3.2). First consider

$$\varphi(z) = \frac{1-z}{1+z}, \qquad \mathrm{Re}\varphi(z) = \frac{1-|z|^2}{|1+z|^2} > 0, \qquad z \in \mathbb{D}.$$

φ is its own inverse:

$$\varphi^{-1} = \varphi : \qquad \varphi(\varphi(z)) = \frac{1 - \dfrac{1-z}{1+z}}{1 + \dfrac{1-z}{1+z}} = z.$$

For $\alpha \in (0,1)$, let $g(z) = z^\alpha$, and define

$$f = \varphi^{-1} \circ g \circ \varphi = \frac{1 - \left(\frac{1-z}{1+z}\right)^\alpha}{1 + \left(\frac{1-z}{1+z}\right)^\alpha}, \qquad \mathbb{D} \to \mathbb{H} \to \mathbb{D}.$$

Then

$$f(0) = 0, \quad f(1) = 1, \quad f'(z) \neq 0.$$

Therefore $f : \mathbb{D} \to \mathbb{D}$ is univalent and $z = 1$ is a fixed point of f. Considering the Taylor expansions

$$(1+z)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \qquad (1-z)^\alpha = \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} z^n,$$

we have

$$f(z) = \frac{(1+z)^\alpha - (1-z)^\alpha}{(1+z)^\alpha + (1-z)^\alpha} = \frac{\sum_{n=0}^{\infty} \binom{\alpha}{2n+1} z^{2n+1}}{\sum_{n=0}^{\infty} \binom{\alpha}{2n} z^{2n}} = \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_{2n+1} z^{2n+1}.$$

Define

$$F(z) = \int_0^z f(w)dw = \int_0^z \sum_{n=0}^{\infty} c_{2n+1} w^{2n+1} dw = \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+2} z^{2n+2}, \qquad z \in \mathbb{D}.$$

By Lemma 3.5, $c_{2n+1} > 0$ for $n \geq 0$. Therefore $|F(z)|$ achieves its maximum on the boundary at $z = 1$:

$$|F(z)| \leq \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+2} |z|^{2n+2} \leq \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+1} |z|^{2n+1} \leq \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+1} = F(1) = \max_{|z|\leq 1} |F(z)|.$$

By the maximal principle, the function

$$\psi(z) = \frac{F(z)}{F(1)}, \qquad z \in \mathbb{D}$$

maps $\mathbb{D}$ into $\mathbb{D}$. Furthermore, $z = 1$ is a fixed point of ψ,

$$\psi(1) = 1, \quad \psi'(1) = \frac{F'(1)}{F(1)} = \frac{f(1)}{F(1)} \neq \infty, \quad \psi''(1) = \frac{f'(1)}{F(1)} = \infty,$$

and

$$\psi^{(k)}(r) = \frac{f^{(k-1)}(r)}{F(1)}, \qquad k \geq 2.$$

Notice that

$$\frac{d}{dz} \left(\frac{1-z}{1+z}\right)^\alpha = \alpha \left(\frac{1-z}{1+z}\right)^{\alpha-1} \frac{-2}{(1+z)^2} = \alpha \left(\frac{1-z}{1+z}\right)^{\alpha-1} \varphi'(z),$$

and

$$\frac{d^2}{dz^2}\left(\frac{1-z}{1+z}\right)^\alpha = \alpha(\alpha-1)\left(\frac{1-z}{1+z}\right)^{\alpha-2}\left(\varphi'(z)\right)^2 + \alpha\left(\frac{1-z}{1+z}\right)^{\alpha-1}\varphi''(z),$$

etc. Using the big O notation for z near 1, we may write

$$\frac{d^k}{dz^k}\left(\frac{1-z}{1+z}\right)^\alpha = k!\binom{\alpha}{k}\left(\frac{1-z}{1+z}\right)^{\alpha-k}\left(\varphi'(z)\right)^k + O\left((1-z)^{\alpha-k+1}\right).$$

Applying the little o notation, we have

$$\begin{aligned}\frac{d^k}{dz^k}g(\varphi(r)) &= \frac{d^k}{dz^k}\left(\frac{1-z}{1+z}\right)^\alpha\Big|_{z=r} \\ &= k!\binom{\alpha}{k}\left(\frac{1-r}{1+r}\right)^{\alpha-k}\left(\frac{-2}{(1+r)^2}\right)^k + O\left((1-r)^{\alpha-k+1}\right) \\ &= k!\binom{\alpha}{k}(1-r)^{\alpha-k}(-1)^k\frac{2^k}{(1+r)^{\alpha+k}} + o\left((1-r)^{\alpha-k}\right)\end{aligned}$$

By Faà di Bruno's formula [7],

$$f^{(\ell)}(r) = \frac{d^\ell}{dz^\ell}\varphi(g(\varphi(r))) = \sum \frac{\ell!}{m_1!\ m_2!\cdots m_\ell!}\varphi^{(m_1+\cdots+m_\ell)}\left(g(\varphi(r)\right)\prod_j\left(\frac{\frac{d^k}{dz^k}g(\varphi(r))}{j!}\right)^{m_j}$$

where the sum is over all ℓ-tuples $(m_1, m_2, \cdots, m_\ell)$ satisfying

$$1m_1 + 2m_2 + \cdots + \ell m_\ell = \ell.$$

Notice that

$$\varphi^{(k)}(z) = 2(-1)^k k!(1+z)^{-(k+1)}, \quad \varphi^{(k)}(g(\varphi(r)) = 2(-1)^k k!\left(1+\left(\frac{1-r}{1+r}\right)^\alpha\right)^{-(k+1)},$$

and

$$\begin{aligned}\prod_j\left(\frac{\frac{d^k}{dz^k}g(\varphi(r))}{j!}\right)^{m_j} &= \prod_j\left\{\binom{\alpha}{j}(1-r)^{\alpha-j}(-1)^j\frac{2^j}{(1+r)^{\alpha+j}} + o\left((1-r)^{\alpha-j}\right)\right\}^{m_j} \\ &= (-1)^{m_1+\cdots+m_\ell}(1-r)^{\alpha-\ell}\left(\frac{2}{1+r}\right)^\ell\prod_j\left\{\binom{\alpha}{j}\frac{(-1)^{j-1}}{(1+r)^\alpha}\right\} \\ &\quad + o\left((1-r)^{\alpha-\ell}\right),\end{aligned}$$

where each term in the product

$$\binom{\alpha}{j}\frac{(-1)^{j-1}}{(1+r)^\alpha} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)(-1)^{j-1}}{j!\ (1+r)^\alpha} = \frac{\alpha(1-\alpha)\cdots(j-1-\alpha)}{j!\ (1+r)^\alpha} > 0$$

for $\alpha \in (0,1)$, and the little o term is obtained by the fact that

$$\lim_{r\to 1} \frac{\prod_j (1-r)^{\alpha-j}}{(1-r)^{\alpha-\ell}} = \lim_{r\to 1} \frac{(1-r)^{(m_1+\cdots+m_\ell)\alpha-\ell}}{(1-r)^{\alpha-\ell}} = \lim_{r\to 1}(1-r)^{(m_1+\cdots+m_\ell-1)\alpha} = 0.$$

Since for given $\alpha \in (0,1)$ and $\ell \geq 1$,

$$\sum \frac{\ell!\ 2(m_1+\cdots+m_\ell)!}{m_1!\ m_2! \cdots m_\ell!} \left(1+\left(\frac{1-r}{1+r}\right)^\alpha\right)^{-(m_1+\cdots+m_\ell+1)}$$

is bounded and > 0 as $r \to 1$, we have

$$f^{(\ell)}(r) = \sum \frac{\ell! 2(m_1+\cdots+m_\ell)!}{m_1!\ m_2! \cdots m_\ell!} \left(1+\left(\frac{1-r}{1+r}\right)^\alpha\right)^{-(m_1+\cdots+m_\ell+1)} (1-r)^{\alpha-\ell} \left(\frac{2}{1+r}\right)^\ell \prod_j \binom{\alpha}{j} \frac{(-1)^{j-1}}{(1+r)^\alpha}$$
$$+ o\left((1-r)^{\alpha-\ell}\right).$$

Consequently

$$\lim_{r\to 1}(1-r)^{\ell-\alpha} f^{(\ell)}(r) = \sum \frac{\ell!\ 2(m_1+\cdots+m_\ell)!}{m_1!\ m_2! \cdots m_\ell!} \prod_j \binom{\alpha}{j} \frac{(-1)^{j-1}}{2^\alpha} = C_{\ell,\alpha} > 0,$$

where $C_{\ell,\alpha}$ is a constant for any given $\alpha \in (0,1)$ and $\ell \geq 1$. Therefore we have

$$\lim_{r\to 1}(1-r)^{n-1-\alpha}\psi^{(n)}(r) = \lim_{r\to 1}(1-r)^{n-1-\alpha} \frac{f^{(n-1)}(r)}{F(1)} = \frac{C_{n-1,\alpha}}{F(1)} > 0, \qquad \forall n \geq 2.$$

We have shown that (3.2) in Proposition 3.2 holds for ψ with $\zeta = 1$. For an arbitrary $\zeta \in \mathbb{T}$, (3.2) is satisfied by $\bar{\zeta}\psi(\zeta z)$.

To prove (3.3), we show that for any $m \geq 1$, there exists a function ψ_m such that ψ_m (or its rotation) satisfies the conditions in (3.3). Let

$$\psi_m(z) = \frac{F_m(z)}{F_m(1)}, \qquad F_j(z) = \int_0^z \frac{F_{j-1}(w)}{F_{j-1}(1)} dw, \qquad 1 \leq j \leq m,$$

where

$$F_1 = F, \qquad F_0 = f, \qquad \psi_1 = \psi$$

are the functions used in the above proof of (3.2). By the construction,

$$
\begin{aligned}
F_1(z) &= \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+2} z^{2n+2} \\
|F_1(z)| &= \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+2} |z|^{2n+2} \le \sum_{n=0}^{\infty} \frac{c_{2n+1}}{2n+2} = F_1(1), \\
F_2(z) &= \int_0^z \frac{F_1(w)}{F_1(1)} dw = \frac{1}{F_1(1)} \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+2)(2n+3)} z^{2n+3}, \\
|F_2(z)| &= \frac{1}{F_1(1)} \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+2)(2n+3)} |z|^{2n+3} \\
&\le \frac{1}{F_1(1)} \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+2)(2n+3)} = F_2(1),
\end{aligned}
$$

etc. For $j = 1, 2, \cdots, m$,

$$
\begin{aligned}
F_j(z) &= \int_0^z \frac{F_{j-1}(w)}{F_{j-1}(1)} dw = \frac{1}{F_{j-1}(1)} \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+2)(2n+3)\cdots(2n+1+j)} z^{2n+1+j}, \\
|F_j(z)| &= \frac{1}{F_{j-1}(1)} \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+2)(2n+3)\cdots(2n+1+j)} |z|^{2n+1+j} \\
&\le \frac{1}{F_{j-1}(1)} \sum_{n=0}^{\infty} \frac{c_{2n+1}}{(2n+2)(2n+3)\cdots(2n+1+j)} = F_j(1).
\end{aligned}
$$

By the maximal principle, the functions

$$
\psi_m(z) = \frac{F_m(z)}{F_m(1)}, \qquad \psi_m(1) = 1
$$

map $\mathbb{D}$ into $\mathbb{D}$. Furthermore,

$$
\begin{aligned}
\psi_m^{(k)}(1) &= \frac{F_m^{(k)}(1)}{F_m(1)} = \frac{F_{m-1}^{(k-1)}(1)}{F_m(1)F_{m-1}(1)} = \frac{F_{m-2}^{(k-2)}(1)}{F_m(1)F_{m-1}(1)F_{m-2}(1)} \\
&= \frac{F_{m-k}(1)}{\prod_{j=0}^{k} F_{m-j}(1)} < \infty, \qquad k = 1, 2, \cdots, m,
\end{aligned}
$$

especially,

$$
\psi_m^{(m)}(1) = \frac{f(1)}{\prod_{j=1}^{m} F_j(1)} < \infty.
$$

Notice that

$$
\psi_m^{(m)}(z) = \frac{\psi(z)}{\prod_{j=1}^{m} F_j(1)} < \infty.
$$

Consequently the proven result (3.2) implies (3.3) for $\zeta = 1$. For an arbitrary $\zeta \in \mathbb{T}$, (3.3) is satisfied by $\bar{\zeta}\psi_m(\zeta z)$.

This completes the proof of Proposition 3.2. □

3.4. **Proof of Theorem 3.3.** We need several lemmas to prove Theorem 3.3.

Lemma 3.6. *If $\varphi(x) = x^2 + ax + b$, then for any $m \geq 0$,*

$$\frac{d^\ell}{dx^\ell} h(\varphi(x)) = \begin{cases} \displaystyle\sum_{j=0}^{m} \frac{(2m)!}{(2j)!(m-j)!} h^{(m+j)}(\varphi)(\varphi')^{2j}, & \ell = 2m; \\ \displaystyle\sum_{j=0}^{m} \frac{(2m+1)!}{(2j+1)!(m-j)!} h^{(m+j+1)}(\varphi)(\varphi')^{2j+1}, & \ell = 2m+1. \end{cases} \tag{3.6}$$

Proof. Again by Faà di Bruno's formula [7],

$$\frac{d^\ell}{dx^\ell} h(\varphi(x)) \quad = \quad \sum \frac{\ell!}{m_1!\ m_2! \cdots m_\ell!} h^{(m_1+\cdots+m_\ell)}(\varphi(x)) \prod_j \left(\frac{\varphi^{(j)}(x)}{j!}\right)^{m_j}$$

where the sum is over all ℓ-tuples $(m_1, m_2, \cdots, m_\ell)$ satisfying

$$1m_1 + 2m_2 + \cdots + \ell m_\ell = \ell.$$

Since $\varphi'' \equiv 2,\ \varphi^{(j)} = 0$ for $j \geq 3$, the product in Faà di Bruno's formula simplifies to

$$\prod_j \left(\frac{\varphi^{(j)}(x)}{j!}\right)^{m_j} = \varphi'(x)^{m_1} \qquad (\textit{with } 0^0 = 1),$$

which implies

$$\frac{d^\ell}{dx^\ell} h(\varphi(x)) = \sum_{m_1+2m_2=\ell} \frac{\ell!}{m_1!\ m_2!} h^{(m_1+m_2)}(\varphi)(\varphi')^{m_1}, \tag{3.7}$$

where

$$m_2 = \begin{cases} m - \frac{m_1}{2}, & \ell = 2m; \\ m - \frac{m_1-1}{2}, & \ell = 2m+1. \end{cases}$$

Relabeling the summation index by j,

$$j = \begin{cases} \frac{m_1}{2}, & \ell = 2m; \\ \frac{m_1-1}{2}, & \ell = 2m+1, \end{cases} \qquad \text{then} \qquad m_1 + m_2 = \begin{cases} m+j, & \ell = 2m; \\ m+j+1, & \ell = 2m+1. \end{cases}$$

Replacing m_1 and m_2 by j and m, (3.7) becomes (3.6).

□

Lemma 3.7. *Let* $f(r) = \dfrac{1}{|r\zeta - \eta|^n},\ r \in [0,1),\ \zeta, \eta \in S^{n-1},\ n \geq 2$. *Let* $\theta_r \in [0,\pi]$ *denote the angle between the* n*-vectors* ζ *and* $r\zeta - \eta$ *for any* $r \in [0,1]$. *Then*

$$f^{(\ell)}(r) = \begin{cases} \dfrac{1}{|r\zeta-\eta|^{n+\ell}} \displaystyle\sum_{j=0}^{m} \frac{(\cos\theta_r)^{2j}(2m)!}{(2j)!(m-j)!} \frac{(-1)^{m+j}\ n(n+2)\cdots(n+2m+2j-2)}{2^{m-j}}, & \ell = 2m; \\ \dfrac{r-\zeta\cdot\eta}{|r\zeta-\eta|^{n+\ell+1}} \displaystyle\sum_{j=0}^{m} \frac{(\cos\theta_r)^{2j}(2m+1)!}{(2j+1)!(m-j)!} \frac{(-1)^{m+j+1}\ n(n+2)\cdots(n+2m+2j)}{2^{m-j}}, & \ell = 2m+1. \end{cases}$$

Proof. Let $h(x) = x^{-n/2},\ \varphi(r) = |r\zeta - \eta|^2$. Then

$$\varphi'(r) = \frac{d}{dr}\sum_{j=1}^{n}(r\zeta_j - \eta_j)^2 = 2(r - \zeta\cdot\eta), \quad \varphi''(r) = 2, \quad \varphi^{(j)}(r) = 0 \ for \ j \geq 3,$$

$$\begin{aligned} h^{(\alpha)}(x) &= \left(-\frac{n}{2}\right)\left(-\frac{n}{2}-1\right)\cdots\left(-\frac{n}{2}-(\alpha-1)\right)x^{-\frac{n}{2}-\alpha} \\ &= (-1)^{\alpha}\frac{n(n+2)\cdots(n+2(\alpha-1))}{2^{\alpha}}\frac{1}{x^{(n+2\alpha)/2}}. \end{aligned}$$

Denote $\zeta = (\zeta_1, \cdots, \zeta_n),\ \eta = (\eta_1, \cdots, \eta_n)$ and $\zeta\cdot\eta = \sum_j \zeta_j\eta_j$ the Euclidean inner product of ζ and η. Notice that

$$(r - \zeta\cdot\eta)^2 = |r\zeta\cdot\zeta - \zeta\cdot\eta|^2 = |\zeta\cdot(r\zeta-\eta)|^2 = |r\zeta-\eta|^2\cos^2\theta_r,$$

therefore

$$\begin{aligned} h^{(m+j)}(\varphi(r))(\varphi'(r))^{2j} &= (-1)^{m+j}\frac{n(n+2)\cdots(n+2(m+j-1))}{2^{m+j}}\frac{2^{2j}(r-\zeta\cdot\eta)^{2j}}{|r\zeta-\eta|^{n+2m+2j}} \\ &= \frac{(\cos\theta_r)^{2j}}{|r\zeta-\eta|^{n+2m}}(-1)^{m+j}\frac{n(n+2)\cdots(n+2m+2j-2)}{2^{m-j}}, \end{aligned}$$

and

$$\begin{aligned} h^{(m+j+1)}(\varphi(r))(\varphi'(r))^{2j+1} &= (-1)^{m+j+1}\frac{n(n+2)\cdots(n+2(m+j))}{2^{m+j+1}}\frac{2^{2j+1}(r-\zeta\cdot\eta)^{2j+1}}{|r\zeta-\eta|^{n+2m+2j+2}} \\ &= \frac{(r\zeta-\eta)(\cos\theta_r)^{2j}}{|r\zeta-\eta|^{n+2m+2}}(-1)^{m+j+1}\frac{n(n+2)\cdots(n+2m+2j)}{2^{m-j}}. \end{aligned}$$

Applying Lemma 3.6 to $f(r) = h(\varphi(r))$, the result of Lemma 3.7 follows. $\square$

Notation. Denote

$$C_{n,k} = (-1)^k 2\left(C(n,k) + kC(n,k-1)\right) \qquad for \quad k \geq 1,\ n \geq 2,$$

where

(3.8)

$$C(n,\ell) = \begin{cases} \displaystyle\sum_{j=0}^{m} \frac{(2m)!}{(2j)!(m-j)!}\frac{(-1)^{m+j}n(n+2)\cdots(n+2m+2j-2)}{2^{m-j}}, & \ell = 2m; \\ \displaystyle\sum_{j=0}^{m} \frac{(2m+1)!}{(2j+1)!(m-j)!}\frac{(-1)^{m+j+1}n(n+2)\cdots(n+2m+2j)}{2^{m-j}}, & \ell = 2m+1; \end{cases}$$

for $m \geq 0$, with $C(n,0) = 1$.

Lemma 3.8. *Let $\zeta, \eta \in S^{n-1}$, $n \geq 2$. Then for $k \geq 1$,*

$$\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right)\right\} = \begin{cases} 0, & \zeta \neq \eta; \\ C_{n,k}, & \zeta = \eta. \end{cases}$$

Proof. Let $g(r) = 1 - r^2$, $f(r) = |r\zeta - \eta|^{-n}$. Then

$$g'(r) = -2r, \quad g''(r) = -2 \quad \textit{and} \quad g^{(j)}(r) \equiv 0, \ j \geq 3.$$

For $k \geq 1$,

$$\begin{aligned}
\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right) &= \frac{d^k}{dr^k}\left(f(r)g(r)\right) = \sum_{j=0}^{k}\binom{k}{j} f^{(k-j)}(r)g^{(j)}(r) \\
&= \binom{k}{0} f^{(k)}(r)g(r) + \binom{k}{1} f^{(k-1)}(r)g'(r) + \binom{k}{2} f^{(k-2)}(r)g''(r) \\
&= (1-r^2)f^{(k)}(r) - 2rkf^{(k-1)}(r) - k(k-1)f^{(k-2)}(r)
\end{aligned}$$

with the convention that $f^{(\alpha)}(r) = g^{(\alpha)}(r) \equiv 0$ for $\alpha < 0$. Let

$$C(n,\ell,\theta) = \begin{cases} \displaystyle\sum_{j=0}^{m}\frac{(\cos\theta)^{2j}(2m)!}{(2j)!(m-j)!}\frac{(-1)^{m+j}n(n+2)\cdots(n+2m+2j-2)}{2^{m-j}}, & \ell = 2m; \\ \displaystyle\sum_{j=0}^{m}\frac{(\cos\theta)^{2j}(2m+1)!}{(2j+1)!(m-j)!}\frac{(-1)^{m+j+1}n(n+2)\cdots(n+2m+2j)}{2^{m-j}}, & \ell = 2m+1; \end{cases}$$

Then

$$C(n,\ell) = C(n,\ell,0), \qquad \ell \geq 1, \ n \geq 2.$$

From Lemma 3.7,

$$f^{(\ell)}(r) = \frac{A(\ell,r,\zeta,\eta)}{|r\zeta-\eta|^{n+\ell}}C(n,\ell,\theta_r), \qquad \textit{where} \qquad A(\ell,r,\zeta,\eta) = \begin{cases} 1, & \ell \textit{ even}; \\ \dfrac{r-\zeta\cdot\eta}{|r\zeta-\eta|}, & \ell \textit{ odd}. \end{cases}$$

Thus

$$\begin{aligned}
&\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right)\right\} \\
= &\lim_{r\to 1}\left\{(1-r)^{n+k-1}\left[(1-r^2)f^{(k)}(r) - 2rkf^{(k-1)}(r) - k(k-1)f^{(k-2)}(r)\right]\right\} \\
= &\lim_{r\to 1}\left\{\frac{(1-r^2)(1-r)^{n+k-1}A(k,r,\zeta,\eta)}{|r\zeta-\eta|^{n+k}}C(n,k,\theta_r)\right. \\
&- \frac{2rk(1-r)^{n+k-1}A(k-1,r,\zeta,\eta)}{|r\zeta-\eta|^{n+k-1}}C(n,k-1,\theta_r) \\
&\left. - \frac{k(k-1)(1-r)^{n+k-1}A(k-2,r,\zeta,\eta)}{|r\zeta-\eta|^{n+k-2}}C(n,k-2,\theta_r)\right\}.
\end{aligned}$$

Notice that $|A(k,r,\zeta,\eta)| \le 1$ is bounded, $\theta_r \to 0$ as $r \to 1$, and $C(n,\ell,0)$ is bounded, hence the last term $\to 0$ as $r \to 1$. In addition,

$$\lim_{r\to 1}\frac{1-r}{|r\zeta-\eta|} = \begin{cases} 0, & \zeta \neq \eta; \\ 1, & \zeta = \eta; \end{cases} \qquad \text{and} \qquad \lim_{r\to 1}\frac{(1-r)A(k,r,\zeta,\eta)}{|r\zeta-\eta|} = \begin{cases} 0, & \zeta \neq \eta; \\ (-1)^k, & \zeta = \eta; \end{cases}$$

so we have

$$\begin{aligned} & \lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right)\right\} \\ = & \lim_{r\to 1}\left\{\frac{(1+r)(1-r)^{n+k}A(k,r,\zeta,\eta)}{|r\zeta-\eta|^{n+k}}C(n,k,\theta_r)\right. \\ & \left. - \frac{2rk(1-r)^{n+k-1}A(k-1,r,\zeta,\eta)}{|r\zeta-\eta|^{n+k-1}}C(n,k-1,\theta_r)\right\} \\ = & \begin{cases} 0, & \zeta \neq \eta; \\ 2(-1)^k C(n,k,0) - 2(-1)^{k-1}kC(n,k-1,0), & \zeta = \eta; \end{cases} \\ = & \begin{cases} 0, & \zeta \neq \eta; \\ 2(-1)^k C(n,k) - 2(-1)^{k-1}kC(n,k-1) = C_{n,k}, & \zeta = \eta. \end{cases} \end{aligned}$$

□

Lemma 3.9. $\qquad C_{n,k} = 2\dfrac{(n+k-2)!}{(n-2)!}$ *for* $k \ge 1,\ n \ge 2.$

We postpone the proof of Lemma 3.9 until after the proof of Theorem 3.3.

Now we prove Theorem 3.3.

Proof. For any $x \in B^n$, $x = r\zeta,\ \zeta \in S^{n-1},\ n \ge 2$, we may write

$$u(r\zeta) = \int_{S^{n-1}} \frac{1-r^2}{|r\zeta-\eta|^n}d\mu(\eta).$$

By Lemma 3.8, for any $k \ge 1$,

$$\int_{S^{n-1}} \lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right)\right\}d\mu(\eta) = \begin{cases} 0, & \mu(\{\zeta\}) = 0; \\ C_{n,k}\ \mu(\{\zeta\}), & \mu(\{\zeta\}) > 0. \end{cases}$$

By the interchangeability of differentiation and integration when the integral of the derivative converges and the Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}
\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}u(r\zeta)\right\} &= \lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}\int_{S^{n-1}}\frac{1-r^2}{|r\zeta-\eta|^n}d\mu(\eta)\right\}\\
&= \lim_{r\to 1}\left\{(1-r)^{n+k-1}\int_{S^{n-1}}\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right)d\mu(\eta)\right\}\\
&= \int_{S^{n-1}}\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}\left(\frac{1-r^2}{|r\zeta-\eta|^n}\right)\right\}d\mu(\eta)\\
&= C_{n,k}\ \mu(\{\zeta\}).
\end{aligned}$$

By Theorem 1.1 in [14] (or by going through the proof of Lemma 3.8 with $k=0$),

$$\lim_{r\to 1}\left\{(1-r)^{n-1}u(r\zeta)\right\} = 2\mu(\{\zeta\}).$$

Thus

$$\lim_{r\to 1}\left\{\frac{(1-r)^{n-1}}{1+r}u(r\zeta)\right\} = \mu(\{\zeta\})$$

and

$$\lim_{r\to 1}\left\{(1-r)^{n+k-1}\frac{d^k}{dr^k}u(r\zeta)\right\} = C_{n,k}\ \lim_{r\to 1}\left\{\frac{(1-r)^{n-1}}{1+r}u(r\zeta)\right\}.$$

So (3.4) holds by Lemma 3.9. Consequently (3.5) holds because each positive harmonic function in the unit ball corresponds to a positive measure on the sphere (ref. [1]), and that the set of non-zero point mass of the measure is countable.

This completes the proof of Theorem 3.3. □

The following is the proof of Lemma 3.9.

Proof. Lemma 3.9 is proved by induction. By the definition,

$$\begin{aligned}
C(n,0) &= 1,\\
C(n,1) &= \frac{(-1)^1\ n}{2^0} = -n,\\
C(n,2) &= \frac{2!\ (-1)^1\ n}{2^1} + \frac{2!\ (-1)^2 n(n+2)}{2!\ 2^0} = -n + n(n+2) = n(n+1).
\end{aligned}$$

For $k=1$ and 2,

$$\begin{aligned}
C_{n,1} &= (-1)^1\ 2(C(n,1)+C(n,0)) = -2(-n+1) = 2\frac{(n+1-2)!}{(n-2)!},\\
C_{n,2} &= (-1)^2\ 2(C(n,2)+2C(n,1)) = 2(n(n+1)-2n) = 2\frac{(n+2-2)!}{(n-2)!}.
\end{aligned}$$

For any k, assuming

$$C_{n,k} = (-1)^k 2\left(C(n,k)+kC(n,k-1)\right) = 2\frac{(n+k-2)!}{(n-2)!},$$

we will prove

$$C_{n,k+1} = (-1)^{k+1} 2\left(C(n,k+1) + (k+1)C(n,k)\right) = 2\frac{(n+k-1)!}{(n-2)!}.$$

Using the induction assumption, the above equation can be written as

$$\begin{aligned}
(-1)^{k+1}\left(C(n,k+1)+(k+1)C(n,k)\right) &= \frac{(n+k-1)(n+k-2)!}{(n-2)!} \\
&= (n+k-1)C_{n,k} \\
&= (n+k-1)(-1)^k\left(C(n,k)+kC(n,k-1)\right),
\end{aligned}$$

so it is sufficient to show

$$-\left(C(n,k+1)+(k+1)C(n,k)\right) = (n+k-1)\left(C(n,k)+kC(n,k-1)\right) \tag{3.9}$$

Write

$$C(n,k) = \sum_j C(n,k,j).$$

Notice that

$$\begin{aligned}
C(n,2m+2,j) &= \frac{(m+1)(2j+1)}{m+1-j}C(n,2m+1,j), \qquad 0 \le j \le m; \\
C(n,2m+2,j+1) &= -\frac{(m+1)(n+2m+2j+2)}{j+1}C(n,2m+1,j), \qquad 0 \le j \le m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
C(n,2m+2) &= \sum_{j=0}^{m+1}\left(\frac{m+1-j}{m+1}+\frac{j}{m+1}\right)C(n,2m+2,j) \\
&= \sum_{j=0}^{m}\frac{m+1-j}{m+1}C(n,2m+2,j) + \sum_{j=1}^{m+1}\frac{j}{m+1}C(n,2m+2,j) \\
&= \sum_{j=0}^{m}\frac{m+1-j}{m+1}C(n,2m+2,j) + \sum_{j=0}^{m}\frac{j+1}{m+1}C(n,2m+2,j+1) \\
&= \sum_{j=0}^{m}(2j+1)C(n,2m+1,j) - \sum_{j=0}^{m}(n+2m+2j+2)C(n,2m+1,j) \\
&= -\sum_{j=0}^{m}(n+2m+1)C(n,2m+1,j) \\
&= -(n+2m+1)C(n,2m+1) \qquad for \quad m \ge 0.
\end{aligned}$$

Similarly,

$$\begin{aligned} C(n,2m+1,j) &= -\frac{(2m+1)(n+2m+2j)}{2j+1}C(n,2m,j), \qquad 0 \le j \le m; \\ C(n,2m+1,j) &= \frac{(2m+1)(2j+2)}{2m-2j}C(n,2m,j+1), \qquad 0 \le j \le m-1. \end{aligned}$$

Consequently,

$$\begin{aligned} C(n,2m+1) &= \sum_{j=0}^{m}\frac{2j+1}{2m+1}C(n,2m+1,j)+\sum_{j=0}^{m-1}\frac{2m-2j}{2m+1}C(n,2m+1,j) \\ &= -\sum_{j=0}^{m}(n+2m+2j)C(n,2m,j)+\sum_{j=0}^{m-1}(2j+2)C(n,2m,j+1) \\ &= -\sum_{j=0}^{m}(n+2m+2j)C(n,2m,j)+\sum_{j=0}^{m}(2j)C(n,2m,j) \\ &= -\sum_{j=0}^{m}(n+2m)C(n,2m,j) \\ &= -(n+2m)C(n,2m) \qquad for \quad m \ge 0. \end{aligned}$$

The above relations between the adjacent $C(n,\ell)$'s can be summarized as

$$C(n,\ell) = -(n+\ell-1)C(n,\ell-1) \qquad for \quad \ell \ge 1.$$

Applying the above equation repeatedly, for any $k \ge 1$ we have

$$\begin{aligned} L.H.S.\ of\ (3.9) &= -C(n,k+1)-(k+1)C(n,k) \\ &= (n+k)C(n,k)-(k+1)C(n,k) = (n-1)C(n,k), \end{aligned}$$

and

$$\begin{aligned} R.H.S.\ of\ (3.9) &= (n+k-1)\left(C(n,k)+kC(n,k-1)\right) \\ &= (n+k-1)C(n,k)+k(n+k-1)C(n,k-1) \\ &= (n+k-1)C(n,k)-kC(n,k) = (n-1)C(n,k). \end{aligned}$$

Therefore (3.9) holds for all $k \ge 1$. This completes the proof of Lemma 3.9. □

Acknowledgments. We thank Pietro Poggi-Corradini for his valuable idea for Proposition 3.2.

4. When is a function not flat?

4.1. **Introduction.** The function

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is well known for its property

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \underline{\text{but}} \qquad f \not\equiv 0.$$

Such a function is called *flat* at the origin. On the other hand, if f is a real analytic function with Taylor expansion on an open interval containing 0, then $f^{(k)}(0) = 0$, $\forall k \geq 0$ implies $f \equiv 0$. The unique continuation problem in PDE is to find conditions such that the solutions of PDE enjoy the same property. There are a large amount of literature in this area originated from the ideas by Carleman [3], called Carleman's method. In this paper we consider the simplest case of one variable.

Theorem 4.1. *Let $f(x) \in C^\infty([a,b])$, $0 \in [a,b]$, and*

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k}}, \quad x \in [a,b] \tag{4.1}$$

for some constant C and $n \geq 2$. Then

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \text{implies} \qquad f \equiv 0 \quad \textit{on} \quad [a,b].$$

From Theorem 4.1 we obtain the following corollary.

Corollary 4.2. *Let $f(x) \in C^\infty([a,b])$, $0 \in [a,b]$, and (4.1) holds for some constant C and $n \geq 2$. Then*

$$f \not\equiv 0 \qquad \text{implies} \qquad \textit{the zero set} \quad \{f^{-1}(0)\} \subset [a,b] \quad \textit{is finite.}$$

An example.

We use an example in [6] to show that the order of singularity in (4.1) is best possible, i.e., there exists a function $f(x) \in C^\infty([-a,a])$, $a > 0$,

$$|f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k+\varepsilon}} \quad for \quad x \in [-a,a] \qquad and \qquad f^{(k)}(0) = 0, \ \forall k \geq 0$$

for some constant C and $\varepsilon > 0$, but $f \not\equiv 0$ on $[-a,a]$.

For $m > 1$, $\varepsilon > 0$, the following equation is considered in [6]:

$$x^2 u''(x) + mxu'(x) - cx^{-\varepsilon}u(x) = 0, \qquad x \in (0,1),$$

or equivalently,

$$u''(x) + \frac{m}{x}u'(x) - \frac{c}{x^{2+\varepsilon}}u(x) = 0, \qquad x \in (0,1), \tag{4.2}$$

— a Bessel differential equation. The general Bessel differential equation takes the form

$$z^2u''(z) + (1-2\alpha)zu'(z) + \left\{\beta^2\gamma^2z^{2\gamma} + (\alpha^2 - \nu^2\gamma^2)\right\}u(z) = 0, \qquad z \in \mathbb{C}. \tag{4.3}$$

It is well known that for (non-integer) $\nu \notin \mathbb{Z}$, the solution for (4.3) is

$$u(z) = z^\alpha \left[\, C_1J_\nu(\beta z^\gamma) + C_2J_{-\nu}(\beta z^\gamma)\,\right],$$

where $C_1,\ C_2$ are arbitrary complex numbers, and J_ν is the Bessel function of order ν, i.e., a solution of equation (4.3) with $\alpha = 0,\ \gamma = 1$. Notice that equation (4.2) with $c > 0$ is equation (4.3) with

$$\alpha = -\frac{m-1}{2}, \quad \beta = i\frac{2\sqrt{c}}{\varepsilon}, \quad \gamma = -\frac{\varepsilon}{2}, \quad \nu = \frac{m-1}{\varepsilon}.$$

By choosing $m > 1,\ \varepsilon \in (0,1)$ such that

$$C_1 = -\,C_2 = -\frac{\pi e^{-i\nu\pi}}{2\,\sin(\nu\pi)}, \qquad \nu = \frac{m-1}{\varepsilon} \notin \mathbb{Z},$$

the solution of equation (4.2) can be written as

$$u(x) = |x|^{-(m-1)/2}K_{(m-1)/\varepsilon}\left(\frac{2\sqrt{c}}{\varepsilon}|x|^{-\varepsilon/2}\right), \qquad x \in (0,1) \tag{4.4}$$

where

$$K_\nu(z) = \frac{\pi}{2}\,\frac{e^{-i\nu\pi}\,(J_{-\nu}(iz) - J_\nu(iz))}{\sin(\nu\pi)}, \qquad \arg z \in (-\pi, \pi/2)$$

is the modified Bessel function of the third kind [19], with the asymptotic property

$$K_\nu(x) \approx \frac{\pi}{2}x^{-1/2}e^{-x} \qquad as \quad x \to +\infty.$$

Therefore in (4.4), the function

$$u(x) \approx \frac{\pi}{2}\left(\frac{2\sqrt{c}}{\varepsilon}\right)^{-1/2} x^{-\frac{m-1}{2}+\frac{\varepsilon}{4}} \exp\left\{-\,\frac{2\sqrt{c}}{\varepsilon}x^{-\varepsilon/2}\right\} \qquad as \quad x \to 0$$

is a nontrivial solution of (4.2) vanishing at $x = 0$ of infinite order. Hence

$$f(x) = u(|x|), \qquad x \in [-a, a], \quad a \in (0,1)$$

is well defined and $f \in \mathcal{C}^\infty([-a,a])$. Taking derivative of equation (4.2) $n-2$ times,

$$\frac{d^{n-2}}{dx^{n-2}}\left\{u''(x) + \frac{m}{x}u'(x) - \frac{c}{x^{2+\varepsilon}}u(x)\right\} = 0, \qquad x \in (0,1),$$

we obtain

$$u^{(n)}(x) + a_{n-1}(x)u^{(n-1)}(x) + \cdots + a_0(x)u(x) = 0, \qquad x \in (0,1).$$

For given $m,\ c$ and ε, the coefficients

$$|a_j(x)| \le C_o\left(\frac{1}{|x|^{n-j+\varepsilon}}\right), \quad j = 0, 1, \cdots, n-1, \quad x \in (0,1)$$

for some constant $C_o > 0$. By the property of $u(x)$ near $x = 0$, we have

$$f^{(n)}(x) + a_{n-1}(x)f^{(n-1)}(x) + \cdots + a_0(x)f(x) = 0, \qquad x \in [-a, a]$$

with

$$f^{(k)}(0) = 0, \ \forall k \geq 0, \qquad |a_j(x)| \leq C\left(\frac{1}{|x|^{n-j+\varepsilon}}\right), \quad j = 0, 1, \cdots, n-1.$$

for some constant $C >$. Thus

$$f^{(k)}(0) = 0, \ \forall k \geq 0, \qquad |f^{(n)}(x)| \leq C \sum_{k=0}^{n-1} \frac{|f^{(k)}(x)|}{|x|^{n-k+\varepsilon}}, \quad x \in [-a, a],$$

and $f \not\equiv 0$ from the non-triviality of u.

Theorem 4.1 leads to applications in ODE as stated in the following two propositions. Based on the above example, the order of the singularity of the coefficients in the assumption of the propositions is sharp.

Proposition 4.3. *Let $f(x) \in \mathcal{C}^\infty$ be a solution of*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = 0, \qquad x \in [-a, a], \ a > 0$$

with

$$|a_k(x)| = O\left(\frac{1}{|x|^{n-k}}\right) \qquad as \quad x \to 0, \qquad k = 0, 1, \cdots, n-1.$$

Then

$$f^{(k)}(0) = 0, \quad \forall k \geq 0 \qquad \Longrightarrow \qquad f \equiv 0 \quad on \quad [-a, a].$$

Proposition 4.4. *Let $f(x), \ g(x) \in \mathcal{C}^\infty$ be solutions of*

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_0(x)y = b(x), \qquad x \in [-a, a], \ a > 0$$

with

$$|a_k(x)| = O\left(\frac{1}{|x|^{n-k}}\right) \qquad as \quad x \to 0, \qquad k = 0, 1, \cdots, n-1.$$

Then

$$f^{(k)}(0) = g^{(k)}(0), \quad \forall k \geq 0 \qquad \Longrightarrow \qquad f \equiv g \quad on \quad [-a, a].$$

4.2. **Proof of Theorem 4.1 and its corollary.** Several lemmas are needed for the proof of Theorem 4.1. The basic idea of the following lemma was considered in [13].

Lemma 4.5. *Let $v(x) \in C^\infty([0, b])$. Assume $v^{(k)}(0) = 0, \ \forall k \geq 0$. Then for $\alpha \geq 1$,*

$$\int_0^b \frac{[v(x)]^2}{x^{\alpha+2}} dx \leq \frac{4}{(\alpha+1)^2} \int_0^b \frac{[v'(x)]^2}{x^\alpha} dx \tag{4.5}$$

Proof. Write $v = v(x), v' = v'(x)$ and so on.

$$\frac{d}{dx}\left(x^{-(\alpha+1)}v^2\right) = -(\alpha+1)x^{-(\alpha+2)}v^2 + x^{-(\alpha+1)}2vv'$$

Since $v^{(k)}(0) = 0$ for $k \geq [\alpha/2]+1$, we have $[v(x)]^2 \sim O(x^{2(k+1)}) = o(x^{\alpha+1})$, thus

$$\int_0^b \frac{d}{dx}\left(x^{-(\alpha+1)}v^2\right)dx = \frac{[v(b)]^2}{b^{\alpha+1}} - \lim_{x\to 0+}\frac{[v(x)]^2}{x^{\alpha+1}} = \frac{[v(b)]^2}{b^{\alpha+1}} \geq 0.$$

Therefore,

$$\begin{aligned}(\alpha+1)\int_0^b \frac{v^2}{x^{\alpha+2}}dx &\leq \int_0^b \frac{2vv'}{x^{\alpha+1}}dx \\ &= \int_0^b 2\left\{\left(\frac{\alpha+1}{2}\right)^{1/2}\frac{v}{x^{(\alpha+2)/2}}\right\}\left\{\left(\frac{2}{\alpha+1}\right)^{1/2}\frac{v'}{x^{\alpha/2}}\right\}dx \\ &\leq \int_0^b \frac{\alpha+1}{2}\frac{v^2}{x^{\alpha+2}}dx + \int_0^b \frac{2}{\alpha+1}\frac{(v')^2}{x^{\alpha}}dx\end{aligned}$$

by applying $2ab \leq a^2+b^2$ to the last inequality. Consequently,

$$\frac{\alpha+1}{2}\int_0^b \frac{v^2}{x^{\alpha+2}}dx \leq \frac{2}{\alpha+1}\int_0^b \frac{(v')^2}{x^{\alpha}}dx,$$

which is equivalent to (4.5). □

Lemma 4.6. *Let $u(x) \in C^\infty([0,b])$. Assume $u^{(k)}(0) = 0$, $\forall k \geq 0$. Then for $\beta \geq 1, n \geq 1$,*

$$\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx \leq \frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^\beta}dx, \qquad for \quad k = 0,\cdots,n-1.$$

Proof. Applying Lemma 4.5 to $v = u^{(n-1-j)}$, $\alpha = \beta+2j$, $\beta \geq 1$, $j = 0,1,\cdots,n-1$, we obtain

$$\int_0^b \frac{\left[u^{(n-1-j)}(x)\right]^2}{x^{\beta+2j+2}}dx \leq \frac{4}{(\beta+2j+1)^2}\int_0^b \frac{[u^{(n-j)}(x)]^2}{x^{\beta+2j}}dx, \quad j = 0,1,\cdots,n-1,$$

or equivalently,

$$\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx \leq \frac{4}{(\beta+2(n-k-1)+1)^2}\int_0^b \frac{[u^{(k+1)}(x)]^2}{x^{\beta+2(n-k-1)}}dx, \quad k = 0,1,\cdots,n-1.$$

Applying the above inequality repeatedly, we have

$$\begin{aligned}\int_0^b \frac{\left[u^{(k)}(x)\right]^2}{x^{\beta+2(n-k)}}dx &\leq \left\{\prod_{m=k}^{n-2}\frac{4}{(\beta+2(n-1-m)+1)^2}\right\}\frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^\beta}dx \\ &\leq \frac{4}{(\beta+1)^2}\int_0^b \frac{[u^{(n)}(x)]^2}{x^\beta}dx,\end{aligned}$$

because $\beta \geq 1,\ \beta + 2(n-m-1) + 1 \geq 2$ for $m = 0, \ldots, n-1$. □

Lemma 4.7. *Let $u(x) \in C^\infty([0,b])$. Assume $u^{(k)}(0) = 0,\ \forall k \geq 0$. Then for $\beta \geq 1, n \geq 1$,*

$$\int_0^b \frac{1}{x^\beta} \sum_{k=0}^{n-1} \left(\frac{u^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_0^b \frac{1}{x^\beta} [u^{(n)}(x)]^2 dx.$$

Proof. Apply Lemma 4.6 to $k = 0, \cdots, n-1$ and sum up both sides. □

Lemma 4.8. *Let $f(x) \in C^\infty([a,b]), 0 \in [a,b]$. Assume $f^{(k)}(0) = 0,\ \forall k \geq 0$. Then for $\beta \geq 1, n \geq 1$,*

$$\int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_a^b \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx.$$

Proof. The function $u(x) = f(-x)$ satisfies the assumptions for Lemma 4.7 on $[0, -a]$, so

$$\int_0^{-a} \frac{1}{x^\beta} \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(-x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_0^{-a} \frac{1}{x^\beta} [f^{(n)}(-x)]^2 dx.$$

Substitute the variable x by $-x$. Since the terms in the sum are of even powers and both sides have the same x^β term, the above inequality can be written as

$$\int_a^0 a\frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \quad \leq \quad \frac{4n}{(\beta+1)^2} \int_a^0 \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx.$$

From Lemma 4.7 the desired inequality is already true for $f(x)$ on $[0,b]$. Combining the results on $[a,0]$ and $[a,b]$, Lemma 4.8 follows. □

The following is the proof of Theorem 4.1.

Proof. $f(x)$ satisfies the assumptions in Lemma 4.8 on $[a,b]$, so for any $\beta \geq 1$,

$$\frac{(\beta+1)^2}{4n} \int_a^b \frac{1}{|x|^\beta} \sum_{k=0}^{n-1} \left(\frac{f^{(k)}(x)}{x^{n-k}} \right)^2 dx \leq \int_a^b \frac{1}{|x|^\beta} [f^{(n)}(x)]^2 dx. \tag{4.6}$$

From (4.1),

$$|f^{(n)}(x)|^2 \leq C^2 \sum_{k=0}^{n-1} \left(\frac{|f^{(k)}(x)|}{|x|^{n-k}} \right)^2$$

thus

$$\int_a^b \frac{1}{|x|^\beta}|f^{(n)}(x)|^2dx \le C^2 \int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{|f^{(k)}(x)|}{|x|^{n-k}}\right)^2 dx. \tag{4.7}$$

Combining (4.6) and (4.7) we have

$$\int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{f^{(k)}(x)}{x^{n-k}}\right)^2 dx \le \frac{4nC^2}{(\beta+1)^2}\int_a^b \frac{1}{|x|^\beta}\sum_{k=0}^{n-1}\left(\frac{|f^{(k)}(x)|}{|x|^{n-k}}\right)^2 dx.$$

If $f \not\equiv 0$ on $[a,b]$, we would have $|f^{(k)}(x)| > 0$ on some sub-interval of $[a,b]$ thus the integrals > 0, which would imply

$$1 \le \frac{4nC^2}{(\beta+1)^2}, \qquad \forall \beta \ge 1 \qquad \Longrightarrow \textit{Contradiction}.$$

Therefore $f(x) \equiv 0$ on $[a.b]$. This completes the proof of Theorem 4.1. □

The proof of Corollary 4.2 follows immediately.

Proof. Under the assumption of Corollary 4.2, $f \not\equiv 0$ implies $f^{(k)}(0) = \beta \ne 0$ for some $k \ge 0$ based on the result of Theorem 4.1. If $\beta > 0$, then $f \in \mathcal{C}^\infty$ yields $f^{(k)}(x) > \beta/2 > 0,\ x \in [-\delta_0, \delta_0] \subset [a,b]$ for some $\delta_0 > 0$. Consequently the k-fold integral

$$f(x) = \int_0^x \cdots \int_0^x f^{(k)}(y)dy \cdots dy > x^k\beta/2 > 0, \qquad 0 < |x| < \delta_0.$$

The case $\beta < 0$ implies $f(x) < 0$ on an open interval containing 0.

For any $x' \in [a,b]$ such that $f(x') = 0$, the condition $f \not\equiv 0$ implies $f(x) \ne 0,\ 0 < |x - x'| < \delta_{x'},\ x \in [a,b]$ for some $\delta_{x'} > 0$. By the compactness of $[a,b]$, the zero set $\{f^{-1}(0)\}$ is at most finite in $[a,b]$. This completes the proof of Corollary 4.2. □

5. Positive measures on the sphere

5.1. **Introduction.** For any positive measure on the unit sphere $S^{n-1} \subset R^n$,

$$u(x) = P[\mu](x) = \int_{S^{n-1}} \frac{1-|x|^2}{|x-\eta|^n} d\mu(\eta)$$

defines a positive harmonic function in the unit ball B^n [1]. From [14], we have

$$\lim_{r\to 1} \frac{(1+r)^{n-1}}{1-r} u(r\zeta) = 2^n \int_{S^{n-1}} \frac{1}{|\zeta-\eta|^n} d\mu(\eta).$$

Consider

$$G(\xi) = \int_{S^{n-1}} \frac{1}{|\xi-\eta|^n} d\mu(\eta), \qquad \xi \in S^{n-1}. \tag{5.1}$$

$G(\xi)$ may be viewed as radial limits of the potential function

$$\int_{S^{n-1}} \frac{1}{|x-\eta|^n} d\mu(\eta), \qquad x \in B^n.$$

We investigate a convergence property of G in Theorem 5.1. Proposition 5.2 and Proposition 5.4 construct measures that induce examples for extreme cases of G. Corollaries 5.3 and 5.5 give the corresponding results for positive harmonic functions.

Theorem 5.1. *For a positive measure μ on the unit sphere S^{n-1}, let*

$$\mathcal{G}_\infty = \{\xi \in S^{n-1} : \ G(\xi) = \infty\}.$$

Then $\mathcal{G}_\infty$ is a dense G_δ-set in the support of μ. In particular, the closure of $\mathcal{G}_\infty$

$$\overline{\mathcal{G}}_\infty = \operatorname{supp}(\mu). \tag{5.2}$$

Proposition 5.2. *There exists a discrete measure $\mu > 0$ such that*

$$G(\xi) < \infty, \qquad \xi \in S^{n-1}$$

almost everywhere with respect to the Lebesgue measure on S^{n-1}.

Corollary 5.3. *There exists a positive harmonic function u defined by a discrete measure on S^{n-1} such that*

$$\lim_{r\to 1} u(r\zeta) = 0, \qquad a.\ e. \quad \zeta \in S^{n-1}$$

with respect to the Lebesgue measure on S^{n-1}.

Proposition 5.4. *There exists a discrete measure $\mu > 0$ such that*

$$G(\xi) = \infty, \quad \forall \xi \in S^{n-1}.$$

Corollary 5.5. *There exists a positive harmonic function u defined by a discrete measure on S^{n-1} such that*

$$\lim_{r\to 1}\frac{u(r\zeta)}{1-r}=\infty, \qquad \forall \zeta\in S^{n-1}.$$

5.2. **Proof of Theorem 5.1.** To prove Theorem 5.1, we need the following three lemmas. The first two are quoted from [1] and [17] without proofs.

Lemma 5.6. (Corollary 6.44 in [1])
Let μ be a positive Borel measure on S^{n-1} and

$$d\mu = d\mu_{a.c.}+d\mu_s = f d\sigma + d\mu_s \tag{5.3}$$

be the Lebesgue decomposition of μ with respect to the Lebesgue measure σ. Then $P[\mu]$ has non-tangential limit $f(\zeta)$ at almost every $\zeta\in S^{n-1}$.

Lemma 5.7. *If $U\subset S^{n-1}$ is an open and $G(\zeta)<\infty$, $\forall\zeta\in U$. Then*

$$\lim_{r\to 1}u(r\zeta)=0, \qquad \forall\zeta\in U.$$

Proof. By the definition of G and [14],

$$G(\zeta)=\lim_{r\to 1}\frac{u(r\zeta)}{1-r^2}. \tag{5.4}$$

Therefore

$$G(\zeta)<\infty, \quad \forall\zeta\in U \quad \Longrightarrow \quad \lim_{r\to 1}u(r\zeta)=0, \quad \forall\zeta\in U.$$

□

The following is the proof of Theorem 5.1.

Proof. First we show that $\mathcal{G}_\infty$ is a G_δ set, i.e., a countable intersection of open sets. For $m=1,2,\cdots$, let

$$G_m(\zeta)=\int_{S^{n-1}}\min\left\{m,\frac{1}{|\zeta-\eta|^n}\right\}d\mu(\eta).$$

The functions $G_m(\zeta)\le m\times\mu\{S^{n-1}\}$ and $G_m(\zeta)$ is continuous in ζ. By Lemma 5.6 and (5.4),

$$G(\zeta)=\sup_m G_m(\zeta)=\lim_{m\to\infty}G_m(\zeta), \qquad \zeta\in S^{n-1}$$

is well defined. Therefore $G(\zeta)$ is lower semi-continuous in ζ. Consequently $\{\zeta\in S^{n-1}:\ G(\zeta)>m\}$ is an open set. By the definition,

$$\mathcal{G}_\infty=\{\xi\in S^{n-1}:\ G(\xi)=\infty\}=\bigcap_{m=1}^{\infty}\left\{\zeta\in S^{n-1}:\ G(\zeta)>m\right\},$$

therefore $\mathcal{G}_\infty$ is a G_δ set.

In the following we prove that $\overline{\mathcal{G}}_\infty = \text{supp}(\mu)$. If $\zeta \notin \text{supp}(\mu)$, then

$$D(\zeta) = \text{distance}\{\zeta,\ \text{supp}(\mu)\} > 0,$$

then

$$G(\zeta) = \int_{S^{n-1} \cap\ \text{supp}(\mu)} \frac{d\mu(\eta)}{|\zeta - \eta|^n} \leq \frac{\mu\{S^{n-1}\}}{D(\zeta)^n} < \infty \qquad \Longrightarrow \qquad \zeta \notin \mathcal{G}_\infty.$$

Hence

$$\mathcal{G}_\infty \subset \text{supp}(\mu) \qquad \textit{and} \qquad \overline{\mathcal{G}}_\infty \subset \overline{\text{supp}(\mu)} = \text{supp}(\mu).$$

To prove $\overline{\mathcal{G}}_\infty = \text{supp}(\mu)$, we show that $\mathcal{G}_\infty$ is dense in $\text{supp}(\mu)$. For any open set $U \subset S^{n-1}$, $U \cap \mathcal{G}_\infty = \emptyset$, we have $f(\zeta) = \lim_{r \to 1} u(r\zeta) = 0$ in (5.3) by applying Lemma 5.7. Therefore

$$\mu_{a.c.}(U) = 0.$$

From [12] and [14],

$$\lim_{r \to 1} (1 - r)^{n-1} u(r\zeta) = 2^n \mu(\{\zeta\}) = 2\mu_s(\{\zeta\}).$$

Consequently

$$\{\zeta \in S^{n-1} :\ \mu_s(\{\zeta\}) > 0\} \ \subset\ \{\zeta \in S^{n-1} :\ \lim_{r \to 1} u(r\zeta) = \infty\}.$$

we obtain

$$\mu_s(U) = 0.$$

Hence

$$\mu(U) = \mu_{a.c.}(U) + \mu_s(U) = 0.$$

We have shown that $U \cap \mathcal{G}_\infty = \emptyset$ implies $mu(U) = 0$, i.e., $\mathcal{G}_\infty$ is dense in $\text{supp}(\mu)$, then (5.2) follows. This concludes the proof of Theorem 5.1.

□

5.3. **Proof of Proposition 5.2.** We state in the following lemma (without proof) a geometric property of Lebesgue measures on the sphere.

Lemma 5.8. *Let σ be the Lebesgue measure on S^{n-1} with $\sigma(S^{n-1}) = 1$. Given $n \geq 2$, there exist a constant $C > 0$ such that*

$$\sigma\{\xi \in S^{n-1} :\ |\xi - \eta| < r\} \leq C r^{n-1}, \qquad \forall \eta \in S^{n-1}.$$

The following is the proof of Proposition 5.2.

Proof. Let $\{\zeta_j\}_{j=1}^{\infty}$ be a dense countable set $\subset S^{n-1}$. Let

$$\mu = \sum_{j=1}^{\infty} \frac{1}{2^j} \delta_{\{\zeta_j\}}$$

be the measure on S^{n-1}. Then

$$G(\xi) = \int_{S^{n-1}} \frac{1}{|\xi - \eta|^n} d\mu(\eta) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{|\xi - \zeta_j|^n}, \qquad \xi \in S^{n-1}.$$

Let

$$A_m = \left\{ \xi \in S^{n-1}, |\xi - \zeta_j| > 2^{-j/3}, j \geq m \right\} \ \setminus \ \bigcup_{k=1}^{m-1} \zeta_k.$$

Then

$$A_m = \bigcap_{j=m}^{\infty} B_j \ \setminus \ \bigcup_{k=1}^{m-1} \zeta_k, \qquad \textit{where} \quad B_j = \left\{ \xi \in S^{n-1}, |\xi - \zeta_j| > 2^{-j/3} \right\},$$

and

$$\begin{aligned} S^{n-1} \ \setminus \ A_m \ &= \ \left(S^{n-1} \setminus \bigcap_{j=m}^{\infty} B_j \right) \bigcup \{\zeta_k\}_{k=1}^{m-1} \\ &= \bigcup_{j=m}^{\infty} \left(S^{n-1} \setminus B_j \right) \bigcup \{\zeta_k\}_{k=1}^{m-1}. \end{aligned}$$

Let σ be the normalized Lebesgue measure on S^{n-1} with $\sigma(S^{n-1}) = 1$. Then $\sigma(\{\zeta_j\}) = 0, \ \forall j$ and by Lemma 5.8,

$$\sigma\left(S^{n-1} \setminus A_m\right) \leq \sum_{j=m}^{\infty} \sigma\left(S^{n-1} \setminus B_j\right) \leq \sum_{j=m}^{\infty} C \left(2^{-j/3}\right)^{n-1}.$$

Hence

$$\sigma\left(S^{n-1} \setminus \bigcup_{m=1}^{\infty} A_m \right) = \sigma\left(\bigcap_{m=1}^{\infty} \left\{ S^{n-1} \setminus A_m \right\} \right) = \lim_{m \to \infty} \sigma\left(S^{n-1} \setminus A_m\right) = 0.$$

Therefore

$$\sigma\left(\bigcup_{m=1}^{\infty} A_m \right) = 1.$$

Notice that, for any $m > 0$,

$$G(\xi) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{|\xi - \zeta_j|^n} \leq \sum_{j=1}^{m-1} \frac{1}{2^j} \frac{1}{|\xi - \zeta_j|^n} + \sum_{j=1}^{\infty} \frac{1}{2^{2j/3}} < \infty, \qquad \forall \xi \in A_m.$$

Consequently,

$$G(\xi) < \infty \qquad a.\ e. \quad in \quad \sigma.$$

□

5.4. **Proof of Proposition 5.4.**

Proof. For any $N > 1$, there exists $X^N = (x_1^N, \cdots, x_N^N) \in S^{n-1} \subset \mathbb{R}^n$ such that

$$E(N) = \sum_{1 \le j \ne k \le N} \frac{1}{\left|x_j^N - x_k^N\right|^n} = \min_{(x_1,\cdots,x_N) \in S^{n-1}} \frac{1}{|x_j - x_k|^n},$$

i.e. X^N is the s-extremal configuration of N points with $s = n$ [8]. For each $N > 2$, define a measure

$$\mu^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\{x_i^N\}}$$

and functions

$$\begin{aligned} \mathcal{U}^N(x) &= N \int_{S^{n-1}} \frac{1}{|x-y|^n} d\mu^N(y) = \sum_{k=1}^{N} \frac{1}{\left|x - x_k^N\right|^n}, \\ \mathcal{U}_i^N(x) &= \mathcal{U}^N(x) - \frac{1}{\left|x - x_i^N\right|^n}, \qquad i = 1, \cdots, N. \end{aligned}$$

By the minimization property of X^N, for any $x \in S^{n-1}$,

$$\begin{aligned} & 2\mathcal{U}_i^N(x) + \sum_{\substack{1 \le j \ne k \le N \\ j,k \ne i}} \frac{1}{\left|x_j^N - x_k^N\right|^n} \\ = & \sum_{\substack{1 \le j \le N \\ j \ne i}} \frac{1}{\left|x_j^N - x\right|^n} + \sum_{\substack{1 \le k \le N \\ k \ne i}} \frac{1}{\left|x - x_k^N\right|^n} + \sum_{\substack{1 \le j \ne k \le N \\ j,k \ne i}} \frac{1}{\left|x_j^N - x_k^N\right|^n} \\ \ge & \; E(N) = \sum_{1 \le j \ne k \le N} \frac{1}{\left|x_j^N - x_k^N\right|^n} \\ = & \; 2\mathcal{U}_i^N(x_i) + \sum_{\substack{1 \le j \ne k \le N \\ j,k \ne i}} \frac{1}{\left|x_j^N - x_k^N\right|^n}. \end{aligned}$$

Consequently

$$\mathcal{U}_i^N(x) \ge \mathcal{U}_i^N(x_i), \quad i = 1, \cdots, N.$$

By the definition of $u_i^N(x)$,

$$\mathcal{U}^N(x) = \mathcal{U}_i^N(x) + \frac{1}{\left|x - x_i^N\right|^n} \quad \Longrightarrow \quad N\mathcal{U}^N(x) = \sum_{i=1}^{N} \mathcal{U}_i^N(x) + \mathcal{U}^N(x).$$

Therefore for $x \in S^{n-1}$,

$$\frac{1}{N}\mathcal{U}^N(x) = \frac{1}{N(N-1)} \sum_{i=1}^{N} \mathcal{U}_i^N(x) \ge \frac{1}{N(N-1)} \sum_{i=1}^{N} \mathcal{U}_i^N(x_i) = \frac{1}{N(N-1)} E(N).$$

For $\varepsilon > 0$, define a measure

$$\mu = C_\varepsilon \sum_{N=2}^{\infty} \frac{1}{N^{1+\varepsilon}} \mu^N, \qquad with \quad C_\varepsilon = \left(\sum_{N=2}^{\infty} \frac{1}{N^{1+\varepsilon}} \right)^{-1}.$$

Then

$$G(\xi) = \int_{S^{n-1}} \frac{1}{|\xi - \eta|^n} d\mu(\eta) = \sum_{N=2}^{\infty} \frac{1}{N^{1+\varepsilon}} \frac{1}{N} \mathcal{U}^N(\xi).$$

By Theorem 2 in [8],

$$E(N) \sim N^{2+\frac{1}{n-1}} \quad as \quad N \to \infty$$

and there exist a constant C (depending on n only) such that

$$E(N) \geq C N^{2+\frac{1}{n-1}}.$$

Consequently, for $x \in S^{n-1}$,

$$\begin{aligned} G(\xi) &= \sum_{N=2}^{\infty} \frac{1}{N^{1+\varepsilon}} \frac{1}{N} \mathcal{U}^N(\xi) \\ &\geq \sum_{N=2}^{\infty} \frac{1}{N^{1+\varepsilon}} \frac{1}{N(N-1)} E(N) \\ &\geq \sum_{N=2}^{\infty} \frac{C N^{2+\frac{1}{n-1}}}{N^{2+\varepsilon}(N-1)} \\ &= C \sum_{N=2}^{\infty} \frac{N^{\frac{1}{n-1}-\varepsilon}}{N-1} = \infty \qquad for \qquad \varepsilon \in \left(0, \frac{1}{n-1}\right). \end{aligned}$$

□

6. On spherical harmonic expansions of complex Borel measures on the unit sphere

6.1. **Introduction.** Let $B = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball in $\mathbb{R}^n$. $S^{n-1} = \partial B$. Let $\mathcal{H}_m(S^{n-1})$ denote the complex vector space of spherical harmonics of degree m. $\mathcal{H}_m(S^{n-1})$ is the restriction to S^{n-1} of the complex vector space $\mathcal{H}_m(\mathbb{R}^n)$ of homogeneous harmonic polynomials in $\mathbb{R}^n$. Let $Z_m(., \eta)$ be the zonal harmonic of $\mathcal{H}_m(S^{n-1})$ with pole η. If μ is a complex measure on S^{n-1}, the spherical harmonic expansion of μ is defined to the series

$$\sum_{m=0}^{\infty} p_m(\zeta),$$

where

$$p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1})$$

for $\zeta \in S^{n-1}$. It is well-known $L^2(S^{n-1}) = \oplus_0^\infty \mathcal{H}_m(S^{n-1})$. Therefore if $f \in L^2(S^{n-1})$, then its spherical harmonic expansion defined as above with $d\mu = f d\sigma$ converges to f in $L^2(S^{n-1})$. It is known [4] that if $1 \le p < 2$ then there is an $\phi \in L^p(S^{n-1})$ whose spherical harmonic expansion diverges almost everywhere. In this paper, however, we will prove that the spherical harmonic expansion of a complex measure enjoys a precise asymptotics. Here is the main result.

Theorem 6.1. *Let μ be a complex Borel measure on the unit sphere S^{n-1} . Let $\sum_{m=0}^{\infty} p_m(\zeta)$ be the spherical harmonic expansion of μ Then*

$$\sum_{m=0}^{N} p_m(\zeta) \sim \frac{2}{(n-1)!} \mu(\{\zeta\}) N^{n-1}$$

as $N \to \infty$.

We have the following corollary, which gives a sufficient condition for the spherical expansion series to diverge.

Corollary 6.2. *Let μ be a complex Borel measure on S^{n-1}. If $\mu(\{\zeta\}) > 0$ for some $\zeta \in S^{n-1}$ then the series*

$$\sum_{m=0}^{\infty} p_m(\zeta)$$

is divergent to $+\infty$. If $\mu(\{\zeta\}) = 0$ then

$$\sum_{m=0}^{N} p_m(\zeta) = o(N^{n-1}).$$

We will construct a positive measure so that its spherical harmonic expansion diverges on a countable dense subset of the unit sphere. The special case of Theorem 6.1 when $n = 2$ implies the following regarding the Fourier series of a complex measure on the unit circle.

Theorem 6.3. *Let μ be a complex Borel measure on S^1. Let*

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

be the Fourier series of μ where a_n, the Fourier coefficient, is defined as

$$a_n = \int_{-\pi}^{\pi} e^{-in\theta} d\mu(e^{i\theta}).$$

Then

$$\sum_{n=-N}^{N} a_n e^{in\theta} \sim 2\mu(\{e^{i\theta}\})N.$$

In particular, if $\mu(\{e^{i\theta}\}) > 0$ then the Fourier series is divergent to $+\infty$, and if $\mu(\{e^{i\theta}\}) = 0$, then

$$\sum_{n=-N}^{N} a_n e^{in\theta} = o(N).$$

Theorem 6.4. *Let μ be a positive measure on S^{n-1}. Let $\sum_{m=0}^{\infty} p_m(\zeta)$ be its spherical harmonic expansion, and let*

$$S_m(\zeta) = \sum_{k=0}^{m} p_k(\zeta).$$

Then for each $\zeta \in S^{n-1}$, the series

$$\sum_{m=0}^{\infty} S_m(\zeta)$$

is Abel summable to $\frac{b(\zeta)}{2^{n-1}}$, where

$$b(\zeta) = \int_{S^{n-1}} \frac{2^n}{|\eta - \zeta|^n} d\mu(\eta)$$

and Also, one has $b(\zeta) \geq \mu(S^{n-1})$ for $\zeta \in S^{n-1}$.

Corollary 6.5. *Let μ be a positive measure on S^1. Let $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ be its Fourier series. The for each $\theta \in [0, 2\pi]$, the series*

$$\sum_{N=0}^{\infty} \left(\sum_{n=-N}^{N} a_n e^{in\theta} \right)$$

is Abel summable to $b(\theta)$, where $b(\theta) \geq \mu(S^{n-1})$as given in Theorem 6.4.

Theorem 6.1 can be restated as that the point mass of a complex measure on the unit sphere can be approximated by spherical harmonics. Here is the result.

Theorem 6.6. *Let μ be a complex Borel measure on the unit sphere S^{n-1}. Then there exist (unique) $p_m(\zeta) \in \mathcal{H}_m(S^{n-1})$ for $m = 0, 1, 2, ...$ such that the point mass of μ is given by, for each $\zeta \in S^{n-1}$,*

$$\mu(\{\zeta\}) = \frac{(n-1)!}{2} \lim_{N\to\infty} \frac{\sum_{m=0}^{N} p_m(\zeta)}{N^{n-1}}.$$

This paper is a continuation of [12].

6.2. **proofs.** Our proof is an application of a well-known Hardy-Littlewood 's Tauberian Theorem (see [10]) through harmonic functions in the unit ball.

Theorem 6.7. *Let $\sum_{m=0}^{\infty} a_m x^m$ converges for $|x| < 1$. Suppose that for some number $\alpha \geq 0$,*

$$f(x) = \sum_{m=0}^{\infty} a_m x^m \sim \frac{A}{(1-x)^{\alpha}} \text{ as } x \nearrow 1$$

(in the sense that $(1-x)^{\alpha} f(x) \to A$), while

$$ma_m \geq -Cm^{\alpha}, m \geq 1.$$

Then

$$\sum_{m=0}^{N} a_m \sim \frac{A}{\Gamma(\alpha+1)} N^{\alpha}.$$

First let us recall that if μ is a complex Borel measure on S^{n-1} and

$$P(x, \eta) = \frac{1-|x|^2}{|x-\eta|^2}$$

is the Poisson kernel of B, then $P[\mu]$ is defined by

$$P[\mu](x) = \int_{S^{n-1}} P(x, \eta) d\mu(\eta).$$

Of course, $P[\mu]$ is harmonic in B. The following is the modification of Corollary 5.34 in [1] adapted to the measure case and is crucial to our method.

Lemma 6.8. *Let μ be a complex measure on S^{n-1} and $u(x) = P[\mu]$. Then there exist (unique) $p_m \in \mathcal{H}(\mathbb{R}^n)$ such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x)$$

for $x \in B$, the series converges absolutely and uniformly on compact subsets of B. Further

$$|p_m(x)| \leq C|\mu|(S^{n-1}) m^{n-2} |x|^m$$

for some positive constant C and $m = 0, 1, 2, ...$ If $x = |x|\zeta$, then $p_m(\zeta)$ is given by

$$p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}).$$

Proof. By Theorem 5.33 of [1], the Poisson kernel can be given by zonal harmonics.

$$P(x,\zeta)=\sum_{m=0}^{\infty} Z_m(x,\zeta)$$

for all $x\in B, \zeta\in S^{n-1}$. The series converges absolutely and uniformly on $K\times S^{n-1}$ for every compact set $K\subset B$. So for any $x\in B$,

$$u(x)=\int_{S^{n-1}} P(x,\zeta)d\mu(\zeta)=\sum_{m=0}^{\infty}\int_{S^{n-1}} P(x,\zeta)Z_m(x,\zeta)d\mu(\zeta).$$

Letting

$$p_m(x)=\int_{S^{n-1}} Z_m(x,\zeta)d\mu(\zeta)$$

for $x\in B$, we notice that $p_m(x)\in\mathcal{H}_m(\mathbb{R}^n)$. On the other hand if $x=|x|\eta$, then $Z_m(x,\zeta)=|x|^m Z_m(\eta,\zeta)$. By [1] (p.95),

$$|Z_m(\eta,\zeta)|\le \dim\mathcal{H}_m(\mathbb{R}^n)$$

Here

$$\dim\mathcal{H}_m(\mathbb{R}^n)=\binom{n+m-1}{n-1}-\binom{n+m-3}{n-1}.$$

By Pascal's triangle, it is equal to

$$\dim\mathcal{H}_m(\mathbb{R}^n)=\binom{n+m-2}{n-2}+\binom{n+m-3}{n-2}.$$

Applying Stirling's formula, one can get $\dim\mathcal{H}_m(\mathbb{R}^n)\sim Cm^{n-2}$ as $m\to\infty$ for a fixed n. Therefore

$$|p_m(x)|\le\int_{S^{n-1}} |Z_m(x,\zeta)|d|\mu|(\zeta)\le C|\mu|(S^{n-1})m^{n-2}|x|^m.$$

□

Here, according to Lemma 6.8, we observe that the spherical expansion of μ is related the harmonic homogeneous polynomial expansion of the harmonic function $P[\mu]$. Therefore, Theorem 6.1 is equivalent to the following

Theorem 6.9. *Let μ be a complex measure on the unit sphere S^{n-1} and $u(x)=P[\mu]$. Write u for $x=r\zeta$, $\zeta\in S^{n-1}$*

$$u(x)=\sum_{m=0}^{\infty} p_m(\zeta)r^m$$

where $p_m(\zeta)\in\mathcal{H}_m(S^{n-1})$. Then

$$\sum_{m=0}^{N} p_m(\zeta)\sim\frac{2}{(n-1)!}\mu(\{\zeta\})N^{n-1}$$

as $N\to\infty$.

Proof. Let $u(x) = P[\mu]$ be the harmonic function defined by the Poisson integral of μ. By [14],

$$\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) = \int_{S^{n-1}} \lim_{r\to 1}(1-r)^{n-1}P(r\zeta,\eta)d\mu(\eta) = 2\int_{S^{n-1}} \delta_\zeta(\eta)d\mu(\eta) = 2\mu(\{\zeta\}).$$

Now we are ready to apply Hardy-Littlewood's Tauberian theorem. First we observe the series

$$\sum_{m=0}^{\infty} p_m(\zeta)r^m$$

is also convergent for $|r| < 1$, by Lemma 6.8. Also by above

$$\sum_{m=0}^{\infty} p_m(\zeta)r^m \sim \frac{2\mu(\{\zeta\})}{(1-r)^{n-1}} \text{ as } r \nearrow 1$$

Taking real and imaginary parts, we have

$$\sum_{m=0}^{\infty} \Re p_m(\zeta)r^m \sim \frac{2\Re\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \text{ as } r \nearrow 1$$

$$\sum_{m=0}^{\infty} \Im p_m(\zeta)r^m \sim \frac{2\Im\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \text{ as } r \nearrow 1.$$

By Lemma 6.8, there exist a positive constant C so that

$$|p_m(\zeta)| \le Cm^{n-2}$$

It follows that

$$m\Re p_m(\zeta) \ge -Cm^{n-1},$$
$$m\Im p_m(\zeta) \ge -Cm^{n-1}.$$

Applying Hardy-Littlewood' theorem with $\alpha = n-1$ we complete the proof. □

Here is the proof of Theorem 6.3.

Proof. In $\mathbb{R}^2$ the zonal functions are given by

$$Z_m(e^{i\theta}, e^{i\phi}) = e^{im(\theta-\phi)} + e^{-im(\theta-\phi)}$$

for $m > 0$, and $Z_0(e^{i\theta}, e^{i\phi}) = 1$. So it follows from Theorem 6.1. □

In order to prove Theorem 6.4. We need the following theorem proved in [12].

Theorem 6.10. *Let u be a positive harmonic function in B. If $0 \le r_1 \le r_2 < 1$, $\omega \in S^{n-1}$, then*

$$\frac{(1-r_2)^{n-1}}{1+r_2}u(r_2\omega) \le \frac{(1-r_1)^{n-1}}{1+r_1}u(r_1\omega),$$
$$\frac{(1+r_2)^{n-1}}{1-r_2}u(r_2\omega) \ge \frac{(1+r_1)^{n-1}}{1-r_1}u(r_1\omega).$$

In particular, if let

$$a(\omega) = \lim_{r\to 1} I(r,\omega),$$
$$b(\omega) = \lim_{r\to 1} J(r,\omega).$$

Then for each $\omega \in S^{n-1}$,

$$a(\omega) = \mu(\{\omega\})$$
$$b(\omega) = \int_{S^{n-1}} f(\eta, \omega) d\mu(\eta)$$

where $f(\eta, \omega)$ *is as defined in Theorem 6.4.*

A series $\sum\limits_{n=0}^{\infty} a_n$ is said to be Abel summable to A if

$$\sum_{n=0}^{\infty} a_n x^n \to A, \text{ as } x \nearrow 1.$$

Now we prove Theorem 6.4.

Proof. Let $u(x) = P[\mu]$, Then u is a positive harmonic function in B. By Theorem 6.10, we see

$$\frac{1}{1-r} u(r\zeta) \to \frac{b(\zeta)}{2^{n-1}}$$

as $r \nearrow 1$, where $b(\zeta)$ is as in Theorem 6.4. But

$$\frac{1}{1-r} u(r\zeta) = \frac{1}{1-r} \sum_{m=0}^{\infty} p_m(\zeta) r^m$$
$$= \sum_{N=0}^{\infty} \left(\sum_{n=0}^{N} p_m(\zeta) \right) r^N.$$

This finishes the proof. □

The following function will produce a positive measure so that its point mass is positive on a countable dense set of S^{n-1}. The function is

$$u(x) = \sum_{k=0}^{\infty} a_i \frac{1 - |x|^2}{|x - \omega_i|^2}$$

where $\sum\limits_{k=0}^{\infty} a_i < \infty$ $(a_i > 0)$ and $\{\omega_i\}_0^{\infty}$ is a countable dense set of S^{n-1}. We note $u(0) = \sum\limits_{k=0}^{\infty} a_i$ and by the Harnack principle, u is a positive harmonic function. It is easy to verify that $a(\omega_i) = a_i$ for $i = 0, 1, 2...$ and otherwise $a(\omega) = 0$. Its measure is given by

$$\mu = \sum_{k=0}^{\infty} a_k \delta_{\omega_k}$$

where δ_ζ is the point measure at ζ.

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