

# ON SPHERICAL HARMONIC EXPANSIONS OF COMPLEX BOREL MEASURES ON THE UNIT SPHERE

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## 1. INTRODUCTION

Let  $B = \{x \in \mathbb{R}^n : |x| < 1\}$  be the unit ball in  $\mathbb{R}^n$ .  $S^{n-1} = \partial B$ . Let  $\mathcal{H}_m(S^{n-1})$  denote the complex vector space of spherical harmonics of degree  $m$ .  $\mathcal{H}_m(S^{n-1})$  is the restriction to  $S^{n-1}$  of the complex vector space  $\mathcal{H}(\mathbb{R}^n)$  of homogeneous harmonic polynomials in  $\mathbb{R}^n$ . Let  $Z_m(\cdot, \eta)$  be the zonal harmonic of  $\mathcal{H}_m(S^{n-1})$  with pole  $\eta$ . If  $\mu$  is a complex measure on  $S^{n-1}$ , the spherical harmonic expansion of  $\mu$  is defined to the series

$$\sum_{m=0}^{\infty} p_m(\zeta),$$

where

$$p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1})$$

for  $\zeta \in S^{n-1}$ . It is well-known  $L^2(S^{n-1}) = \bigoplus_0^{\infty} \mathcal{H}_m(S^{n-1})$ . Therefore if  $f \in L^2(S^{n-1})$ , then its spherical harmonic expansion defined as above with  $d\mu = f d\sigma$  converges to  $f$  in  $L^2(S^{n-1})$ . It is known [4] that if  $1 \leq p < 2$  then there is an  $\phi \in L^p(S^{n-1})$  whose spherical harmonic expansion diverges almost everywhere. In this paper, however, we will prove that the spherical harmonic expansion of a complex measure enjoys a precise asymptotics. Here is the main result.

**Theorem 1.1.** *Let  $\mu$  be a complex Borel measure on the unit sphere  $S^{n-1}$ . Let  $\sum_{m=0}^{\infty} p_m(\zeta)$  be the spherical harmonic expansion of  $\mu$ . Then*

$$\sum_{m=0}^N p_m(\zeta) \sim \frac{2}{(n-1)!} \mu(\{\zeta\}) N^{n-1}$$

as  $N \rightarrow \infty$ .

We have the following corollary, which gives a sufficient condition for the spherical expansion series to diverge.

**Corollary 1.2.** *Let  $\mu$  be a complex Borel measure on  $S^{n-1}$ . If  $\mu(\{\zeta\}) > 0$  for some  $\zeta \in S^{n-1}$  then the series*

$$\sum_{m=0}^{\infty} p_m(\zeta)$$

is divergent to  $+\infty$ . If  $\mu(\{\zeta\}) = 0$  then

$$\sum_{m=0}^N p_m(\zeta) = o(N^{n-1}).$$

We will construct a positive measure so that its spherical harmonic expansion diverges on a countable dense subset of the unit sphere. The special case of Theorem 1. 1 when  $n = 2$  implies the following regarding the Fourier series of a complex measure on the unit circle.

**Theorem 1.3.** *Let  $\mu$  be a complex Borel measure on  $S^1$ . Let*

$$\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$$

*be the Fourier series of  $\mu$  where  $a_n$ , the Fourier coefficient, is defined as*

$$a_n = \int_{-\pi}^{\pi} e^{-in\theta} d\mu(e^{i\theta}).$$

*Then*

$$\sum_{n=-N}^N a_n e^{in\theta} \sim 2\mu(\{e^{i\theta}\})N.$$

*In particular, if  $\mu(\{e^{i\theta}\}) > 0$  then the Fourier series is divergent to  $+\infty$ , and if  $\mu(\{e^{i\theta}\}) = 0$ , then*

$$\sum_{n=-N}^N a_n e^{in\theta} = o(N).$$

**Theorem 1.4.** *Let  $\mu$  be a positive measure on  $S^{n-1}$ . Let  $\sum_{m=0}^{\infty} p_m(\zeta)$  be its spherical harmonic expansion, and let*

$$S_m(\zeta) = \sum_{k=0}^m p_k(\zeta).$$

*Then for each  $\zeta \in S^{n-1}$ , the series*

$$\sum_{m=0}^{\infty} S_m(\zeta)$$

*is Abel summable to  $\frac{b(\zeta)}{2^{n-1}}$ , where*

$$b(\zeta) = \int_{S^{n-1}} f(\eta, \zeta) d\mu(\eta)$$

*and*

$$f(\eta, \zeta) = \begin{cases} \infty & \eta = \zeta \\ \frac{2^n}{|\eta - \zeta|^n} & \eta \neq \zeta. \end{cases}$$

*Also, one has  $b(\zeta) \geq \mu(S^{n-1})$  for  $\zeta \in S^{n-1}$ .*

**Corollary 1.5.** *Let  $\mu$  be a positive measure on  $S^1$ . Let  $\sum_{n=-\infty}^{\infty} a_n e^{in\theta}$  be its Fourier series. The for each  $\theta \in [0, 2\pi]$ , the series*

$$\sum_{N=0}^{\infty} \left( \sum_{n=-N}^N a_n e^{in\theta} \right)$$

*is Abel summable to  $b(\theta)$ , where  $b(\theta) \geq \mu(S^{n-1})$ .*

Theorem 1.1 can be restated as that the point mass of a complex measure on the unit sphere can be approximated by spherical harmonics. Here is the result.

**Theorem 1.6.** *Let  $\mu$  be a complex Borel measure on the unit sphere  $S^{n-1}$ . Then there exist (unique)  $p_m(\zeta) \in \mathcal{H}_m(S^{n-1})$  for  $m = 0, 1, 2, \dots$  such that the point mass of  $\mu$  is given by, for each  $\zeta \in S^{n-1}$ ,*

$$\mu(\{\zeta\}) = \frac{(n-1)!}{2} \lim_{N \rightarrow \infty} \frac{\sum_{m=0}^N p_m(\zeta)}{N^{n-1}}.$$

This paper is a continuation of [3].

## 2. PROOFS

Our proof is an application of a well-known Hardy-Littlewood's Tauberian Theorem (see [2]) through harmonic functions in the unit ball.

**Theorem 2.1.** *Let  $\sum_{m=0}^{\infty} a_m x^m$  converges for  $|x| < 1$ . Suppose that for some number  $\alpha \geq 0$ ,*

$$f(x) = \sum_{m=0}^{\infty} a_m x^m \sim \frac{A}{(1-x)^\alpha} \text{ as } x \nearrow 1$$

*(in the sense that  $(1-x)^\alpha f(x) \rightarrow A$ ), while*

$$m a_m \geq -C m^\alpha, m \geq 1.$$

*Then*

$$\sum_{m=0}^N a_m \sim \frac{A}{\Gamma(\alpha+1)} N^\alpha.$$

First let us recall that if  $\mu$  is a complex Borel measure on  $S^{n-1}$  and

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^2}$$

is the Poisson kernel of  $B$ , then  $P[\mu]$  is defined by

$$P[\mu](x) = \int_{S^{n-1}} P(x, \eta) d\mu(\eta).$$

Of course,  $P[\mu]$  is harmonic in  $B$ . The following is the modification of Corollary 5.34 in [1] adapted to the measure case and is crucial to our method.

**Lemma 2.2.** *Let  $\mu$  be a complex measure on  $S^{n-1}$  and  $u(x) = P[\mu]$ . Then there exist (unique)  $p_m \in \mathcal{H}(\mathbb{R}^n)$  such that*

$$u(x) = \sum_{m=0}^{\infty} p_m(x)$$

for  $x \in B$ , the series converges absolutely and uniformly on compact subsets of  $B$ . Further

$$|p_m(x)| \leq C|\mu|(S^{n-1})m^{n-2}|x|^m$$

for some positive constant  $C$  and  $m = 0, 1, 2, \dots$ . If  $x = |x|\zeta$ , then  $p_m(\zeta)$  is given by

$$p_m(\zeta) = \int_{S^{n-1}} Z_m(\zeta, \eta) d\mu(\eta) \in \mathcal{H}_m(S^{n-1}).$$

*Proof.* By Theorem 5.33 of [1], the Poisson kernel can be given by zonal harmonics.

$$P(x, \zeta) = \sum_{m=0}^{\infty} Z_m(x, \zeta)$$

for all  $x \in B, \zeta \in S^{n-1}$ . The series converges absolutely and uniformly on  $K \times S^{n-1}$  for every compact set  $K \subset B$ . So for any  $x \in B$ ,

$$u(x) = \int_{S^{n-1}} P(x, \zeta) d\mu(\zeta) = \sum_{m=0}^{\infty} \int_{S^{n-1}} P(x, \zeta) Z_m(x, \zeta) d\mu(\zeta).$$

Letting

$$p_m(x) = \int_{S^{n-1}} Z_m(x, \zeta) d\mu(\zeta)$$

for  $x \in B$ , we notice that  $p_m(x) \in \mathcal{H}_m(\mathbb{R}^n)$ . On the other hand if  $x = |x|\eta$ , then  $Z_m(x, \zeta) = |x|^m Z_m(\eta, \zeta)$ . By [1] (p.95),

$$|Z_m(\eta, \zeta)| \leq \dim \mathcal{H}_m(\mathbb{R}^n)$$

Here

$$\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}.$$

By Pascal's triangle, it is equal to

$$\dim \mathcal{H}_m(\mathbb{R}^n) = \binom{n+m-2}{n-2} + \binom{n+m-3}{n-2}.$$

Applying Stirling's formula, one can get  $\dim \mathcal{H}_m(\mathbb{R}^n) \sim Cm^{n-2}$  as  $m \rightarrow \infty$  for a fixed  $n$ . Therefore

$$|p_m(x)| \leq \int_{S^{n-1}} |Z_m(x, \zeta)| d|\mu|(\zeta) \leq C|\mu|(S^{n-1})m^{n-2}|x|^m.$$

■

Here, according to Lemma 2.2, we observe that the spherical expansion of  $\mu$  is related the harmonic homogeneous polynomial expansion of the harmonic function  $P[\mu]$ . Therefore, Theorem 1.1 is equivalent to the following

**Theorem 2.3.** *Let  $\mu$  be a complex measure on the unit sphere  $S^{n-1}$  and  $u(x) = P[\mu]$ . Write  $u$  for  $x = r\zeta$ ,  $\zeta \in S^{n-1}$*

$$u(x) = \sum_{m=0}^{\infty} p_m(\zeta)r^m$$

where  $p_m(\zeta) \in \mathcal{H}_m(S^{n-1})$ . Then

$$\sum_{m=0}^N p_m(\zeta) \sim \frac{2}{(n-1)!} \mu(\{\zeta\}) N^{n-1}$$

as  $N \rightarrow \infty$ .

*Proof.* Let  $u(x) = P[\mu]$  be the harmonic function defined by the Poisson integral of  $\mu$ . We first observe as in [1] and [3] that

$$(1-r)^{n-1}u(r\zeta) \rightarrow 2\mu(\{\zeta\})$$

as  $r \rightarrow 1$ . Indeed, for  $0 \leq r < 1$

$$(1-r)^{n-1}u(r\zeta) = \int_{S^{n-1}} (1-r)^{n-1}P(r\zeta, \eta)d\mu(\eta)$$

But it is easy to see

$$|(1-r)^{n-1}P(r\zeta, \eta)| \leq 2,$$

and

$$\lim_{r \rightarrow 1} (1-r)^{n-1}P(r\zeta, \eta) = 2\delta_\zeta(\eta),$$

where

$$\delta_\zeta(\eta) = \begin{cases} 1 & \eta = \zeta \\ 0 & \eta \neq \zeta \end{cases}.$$

By the Dominated Convergence Theorem, it follows

$$\lim_{r \rightarrow 1} (1-r)^{n-1}u(r\zeta) = \int_{S^{n-1}} \lim_{r \rightarrow 1} (1-r)^{n-1}P(r\zeta, \eta)d\mu(\eta) = 2 \int_{S^{n-1}} \delta_\zeta(\eta)d\mu(\eta) = 2\mu(\{\zeta\}).$$

Now we are ready to apply Hardy-Littlewood's Tauberian theorem. First we observe the series

$$\sum_{m=0}^{\infty} p_m(\zeta)r^m$$

is also convergent for  $|r| < 1$ , by Lemma 2.2. Also by above

$$\sum_{m=0}^{\infty} p_m(\zeta)r^m \sim \frac{2\mu(\{\zeta\})}{(1-r)^{n-1}} \text{ as } r \nearrow 1$$

Taking real and imaginary parts, we have

$$\sum_{m=0}^{\infty} \Re p_m(\zeta)r^m \sim \frac{2\Re\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \text{ as } r \nearrow 1$$

$$\sum_{m=0}^{\infty} \Im p_m(\zeta) r^m \sim \frac{2\Im\{\mu(\{\zeta\})\}}{(1-r)^{n-1}} \text{ as } r \nearrow 1.$$

By Lemma 2.2, there exist a positive constant  $C$  so that

$$|p_m(\zeta)| \leq C m^{n-2}$$

It follows that

$$m\Re p_m(\zeta) \geq -C m^{n-1},$$

$$m\Im p_m(\zeta) \geq -C m^{n-1}.$$

Applying Hardy-Littlewood' theorem with  $\alpha = n - 1$  we complete the proof.  $\blacksquare$

Here is the proof of Theorem 1.3.

*Proof.* In  $\mathbb{R}^2$  the zonal functions are given by

$$Z_m(e^{i\theta}, e^{i\phi}) = e^{im\theta} + e^{-im\phi}$$

for  $m > 0$ , and  $Z_0(e^{i\theta}, e^{i\phi}) = 1$ . So it follows from Theorem 1.1.  $\blacksquare$

In order to prove Theorem 1.4. We need the following theorem proved in [3].

**Theorem 2.4.** *Let  $u$  be a positive harmonic function in  $B$ . If  $0 \leq r_1 \leq r_2 < 1$ ,  $\omega \in S^{n-1}$ , then*

$$\frac{(1-r_2)^{n-1}}{1+r_2} u(r_2\omega) \leq \frac{(1-r_1)^{n-1}}{1+r_1} u(r_1\omega),$$

$$\frac{(1+r_2)^{n-1}}{1-r_2} u(r_2\omega) \geq \frac{(1+r_1)^{n-1}}{1-r_1} u(r_1\omega).$$

In particular, if let

$$a(\omega) = \lim_{r \rightarrow 1} I(r, \omega),$$

$$b(\omega) = \lim_{r \rightarrow 1} J(r, \omega).$$

Then for each  $\omega \in S^{n-1}$ ,

$$a(\omega) = \mu(\{\omega\})$$

$$b(\omega) = \int_{S^{n-1}} f(\eta, \omega) d\mu(\eta)$$

where  $f(\eta, \omega)$  is as defined in Theorem 1.4.

A series  $\sum_{n=0}^{\infty} a_n$  is said to be Abel summable to  $A$  if

$$\sum_{n=0}^{\infty} a_n x^n \rightarrow A, \text{ as } x \nearrow 1.$$

Now we prove Theorem 1.4.

*Proof.* Let  $u(x) = P[\mu]$ , Then  $u$  is a positive harmonic function in  $B$ . By Theorem 2.4, we see

$$\frac{1}{1-r}u(r\zeta) \rightarrow \frac{b(\zeta)}{2^{n-1}}$$

as  $r \nearrow 1$ , where  $b(\zeta)$  is as in Theorem 1.4. But

$$\begin{aligned} \frac{1}{1-r}u(r\zeta) &= \frac{1}{1-r} \sum_{m=0}^{\infty} p_m(\zeta)r^m \\ &= \sum_{N=0}^{\infty} \left( \sum_{n=0}^N p_m(\zeta) \right) r^N. \end{aligned}$$

This finishes the proof. ■

The following function will produce a positive measure so that its point mass is positive on a countable dense set of  $S^{n-1}$ . The function is

$$u(x) = \sum_{k=0}^{\infty} a_k \frac{1 - |x|^2}{|x - \omega_k|^2}$$

where  $\sum_{k=0}^{\infty} a_k < \infty$  ( $a_i > 0$ ) and  $\{\omega_i\}_0^{\infty}$  is a countable dense set of  $S^{n-1}$ . We note

$u(0) = \sum_{k=0}^{\infty} a_k$  and by the Harnack principle,  $u$  is a positive harmonic function. It is easy to verify that  $a(\omega_i) = a_i$  for  $i = 0, 1, 2, \dots$  and otherwise  $a(\omega) = 0$ . Its measure is given by

$$\mu = \sum_{k=0}^{\infty} a_k \delta_{\omega_k}$$

where  $\delta_{\zeta}$  is the point measure at  $\zeta$ .

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