On the Monotonicity of Positive Invariant Harmonic Functions in the Unit Ball

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Yifei Pan
Department of Mathematical Sciences
Indiana University - Purdue University Fort Wayne
Fort Wayne, IN 46805-1499
School of Mathematics and Informatics
Jiangxi Normal University, Nanchang, China
email: pan@ipfw.edu

Mei Wang
Department of Statistics
University of Chicago
Chicago, IL 60637
email: meiwang@galton.uchicago.edu

ON THE MONOTONICITY OF POSITIVE INVARIANT HARMONIC FUNCTIONS IN THE UNIT BALL

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1. Introduction

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\},\ n \geq 2$ be the open unit ball in $\mathbb{R}^n$. $S^{n-1} = \partial B^n$. Consider the differential operator

$$\Delta_\lambda = (1-|x|^2)\left\{\frac{1-|x|^2}{4}\sum_j \frac{\partial^2}{\partial x_j^2} + \lambda \sum_j x_j \frac{\partial}{\partial x_j} + \lambda(\frac{n}{2} - 1 - \lambda)\right\},\ \lambda \in \mathbb{R}.$$

In this paper, we prove a monotonicity property of invariant harmonic functions that are solutions of $\Delta_\lambda u = 0$ and are defined by positive Borel measures on the sphere with respect to the Poisson kernel P_λ (see below).

This section describes the theorems and their corollaries. The proofs are provided in the next two sections.

Theorem 1.1. *Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. For $\zeta \in S^{n-1}$, if $\lambda > -\frac{n}{2}$ (if $\lambda < -\frac{n}{2}$), the function*

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}} u(r\zeta)$$

is decreasing (increasing) for $0 \leq r < 1$, and the function

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} u(r\zeta)$$

is increasing (decreasing) for $0 \leq r < 1$. Also

$$\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) = \begin{cases} 2^{1+2\lambda}\mu(\{\zeta\}), & \lambda > -\frac{n}{2} \\ \infty, & \lambda < -\frac{n}{2},\ \mu(\{\zeta\}^c) > 0 \\ 2^{1+2\lambda}\mu(\{\zeta\}), & \lambda < -\frac{n}{2},\ \mu(\{\zeta\}^c) = 0 \end{cases}$$

and

$$\lim_{r\to 1}\frac{u(r\zeta)}{(1-r)^{1+2\lambda}} = \int_{S^{n-1}} \frac{2^{1+2\lambda}}{|\zeta - \xi|^{n+2\lambda}} d\mu(\xi).$$

Remarks.

(1) Invariant harmonic functions are the solutions of $\Delta_\lambda u = 0$. These solutions also satisfy certain invariance property with respect to Möbius transformation. Invariant harmonic functions generally do not possess good boundary regularity, as shown in Liu and Peng [3].

(2) Let μ be a positive Borel measure on S^{n-1} and P_λ be the Poisson kernel

$$P_\lambda(x,\zeta) = \frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}.$$

It is known that the integral

$$u(x) = \int_{S^{n-1}} P_\lambda(x,\zeta)d\mu(\zeta)$$

defines an invariant harmonic function in B^n ([1], p. 119).

(3) On the completion of the current work, we learned that the limit cases for $n=2, \lambda=0$ in Theorem 1.1 were obtained by Simon and Wolff ([7], ref. Chapter 10, p. 546 in [6]).

(4) The critical value $\lambda = -\frac{n}{2}$ yields the degenerate case with the constant Poisson kernel.

The following theorem characterizes the behavior of invariant harmonic functions on the rays.

Theorem 1.2. *Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. Let $\zeta \in S^{n-1}$ and $0 \le r' \le r < 1$.*

If $\lambda > -\frac{n}{2}$,

$$\left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1} \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1} u(r'\zeta)$$

If $\lambda < -\frac{n}{2}$,

$$\left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1} u(r'\zeta)$$

For $r'=0$, the above becomes

$$\frac{(1-r)^{1+2\lambda}}{(1+r)^{n-1}}u(0) \le u(r\zeta) \le \frac{(1+r)^{1+2\lambda}}{(1-r)^{n-1}}u(0)$$

for $\lambda > -\frac{n}{2}$, and

$$\frac{(1+r)^{1+2\lambda}}{(1-r)^{n-1}}u(0) \le u(r\zeta) \le \frac{(1-r)^{1+2\lambda}}{(1+r)^{n-1}}u(0)$$

for $\lambda < -\frac{n}{2}$.

Remark. Case $\lambda = 0$ is the classical Harnack Inequality in B^n.

Corollary 1.3. *Let U be the potential function defined in B^n by a positive Borel measure μ on S^{n-1} as follows:*

$$U(x) = \int_{S^{n-1}} \frac{1}{|x-\eta|^{n+2\lambda}} d\mu(\eta).$$

For $\zeta \in S^{n-1}$, if $\lambda > -\frac{n}{2}$ (if $\lambda < -\frac{n}{2}$), the function

$$(1-r)^{n+2\lambda} U(r\zeta)$$

is decreasing (increasing) for $0 \le r < 1$.

In Theorem 1.1, $\lambda = \frac{n}{2} - 1$ corresponds to the Laplace-Beltrami operator $\Delta_{n/2-1}$ and the Poincaré metric. It is known([2]) that given a positive invariant harmonic function (solutions of $\Delta_{n/2-1} u = 0$), there exists a positive Borel measure μ on S^{n-1}, such that

$$u(x) = \int_{S^{n-1}} P_{n/2-1}(x,\zeta) d\mu(\zeta)$$

In this case the monotonicity property in Theorem 1.1 implies the following corollary.

Corollary 1.4. *Let u be a positive solution of $\Delta_{n/2-1} u = 0$ in B^n. Then*

$$\left(\frac{1-r}{1+r}\right)^{n-1} u(r\zeta) \text{ decreasing in } r,$$

$$\left(\frac{1+r}{1-r}\right)^{n-1} u(r\zeta) \text{ increasing in } r.$$

Corollary 1.5. *Let u be a positive harmonic function with respect to the Laplace operator ($\lambda = 0$) defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson Kernel P_0. For $\zeta \in S^{n-1}$, $0 \le r < 1$, the function*

$$\frac{(1-r)^{n-1}}{1+r} u(r\zeta)$$

is decreasing and the function

$$\frac{(1+r)^{n-1}}{1-r} u(r\zeta)$$

is increasing. In addition,

$$\lim_{r\to 1} (1-r)^{n-1} u(r\zeta) = 2\mu(\{\zeta\}),$$

$$\lim_{r\to 1} \frac{u(r\zeta)}{1-r} = \int_{S^{n-1}} \frac{2}{|\zeta-\xi|^n} d\mu(\xi).$$

Corollary 1.5 is the same as a result in [4].

Corollary 1.6. *Let $B^n(R)$ be the open ball of radius R. Let $u(z)$ be an invariant harmonic function in $B^n(R)$ (a.k.a $u(Rz)$ is invariant harmonic in B^n) defined by the Poisson kernel $P_\lambda(\frac{x}{R},\zeta)$. For $\zeta \in S^{n-1}$, if $\lambda > -\frac{n}{2}$ (if $\lambda < -\frac{n}{2}$), the function*

$$\frac{1}{R^{n-2-2\lambda}} \frac{(R-r)^{n-1}}{(R+r)^{1+2\lambda}} u(r\zeta)$$

is decreasing (increasing) and the function

$$\frac{1}{R^{n-2-2\lambda}} \frac{(R+r)^{n-1}}{(R-r)^{1+2\lambda}} u(r\zeta)$$

is increasing (decreasing) in r for $0 \le r < R$. The case $\lambda = 0$ gives the monotonicity of functions

$$\left(\frac{R-r}{R}\right)^{n-2} \frac{R-r}{R+r} u(r\zeta) \qquad \text{and} \qquad \left(\frac{R+r}{R}\right)^{n-2} \frac{R+r}{R-r} u(r\zeta),$$

which implies that, $\forall x \in B^n(r), 0 \le r < R$,

$$\left(\frac{R}{R+r}\right)^{n-2} \frac{R-r}{R+r} u(0) \le u(x) \le \left(\frac{R}{R-r}\right)^{n-2} \frac{R+r}{R-r} u(0)$$

— the classical Harnack Inequality.

Corollary 1.7. *Let u be a positive invariant harmonic function defined in B^n by a positive Borel measure μ on S^{n-1} with the Poisson kernel P_λ. Let $0 \le r' \le r < 1$. If $\lambda > -\frac{n}{2}$,*

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}} \max_{|x|=r} u(x) \le \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}} \max_{|x|=r'} u(x)$$

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} \min_{|x|=r} u(x) \ge \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}} \min_{|x|=r'} u(x)$$

If $\lambda < -\frac{n}{2}$,

$$\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}} \max_{|x|=r} u(x) \le \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}} \max_{|x|=r'} u(x)$$

$$\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}} \min_{|x|=r} u(x) \ge \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}} \min_{|x|=r'} u(x)$$

Similar results are obtained in complex space $\mathbb{C}^n$. Let

$$P_\alpha(z,\zeta) = \frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}, \ \alpha \in \mathbb{R}$$

be the Poisson-Szegö kernel for the operator

$$\Delta_{\alpha,\beta} = 4(1-|z|^2)\left\{\sum_{i,j}(\delta_{i,j} - z_i\overline{z}_j)\frac{\partial^2}{\partial z_i \partial \overline{z}_j} + \alpha\sum_j z_j\frac{\partial}{\partial z_j} + \beta\sum_j \overline{z}_j\frac{\partial}{\partial \overline{z}_j} - \alpha\beta\right\}$$

with $\alpha = \beta$, where $z\cdot\overline{\zeta} = \sum_{i=1}^n z_i\overline{\zeta}_i$. Define

$$u(z) = \int_{S^{n-1}} P_\alpha(z,\zeta)d\mu(\zeta), \quad \alpha\in\mathbb{R}.$$

Theorem 1.8. *Let u be a positive invariant harmonic function defined in the unit ball $B^n\subset\mathbb{C}^n$ by a positive Borel measure μ on $S^{n-1} = \partial B^n$ with the Poisson-Szegö kernel. Given $\zeta\in S^{n-1}$, if $\alpha > -n$ (if $\alpha < -n$), the function*

$$\frac{(1-r)^n}{(1+r)^{n+2\alpha}}u(r\zeta)$$

is decreasing (increasing) for $0\le r<1$, and the function

$$\frac{(1+r)^n}{(1-r)^{n+2\alpha}}u(r\zeta)$$

is increasing (decreasing) for $0\le r<1$. Also

$$\lim_{r\to 1}(1-r)^n u(r\zeta) = \begin{cases} 2^{n+2\alpha}\mu(\{\zeta\}), & \alpha > -n \\ \infty, & \alpha < -n,\ \mu(\{\zeta\}^c) > 0 \\ 2^{n+2\alpha}\mu(\{\zeta\}), & \alpha < -n,\ \mu(\{\zeta\}^c) = 0 \end{cases}$$

and

$$\lim_{r\to 1}\frac{u(r\zeta)}{(1-r)^{n+2\alpha}} = \int_{S^{n-1}}\frac{2^{n+2\alpha}}{|\zeta-\eta|^{2n+2\alpha}}d\mu(\eta).$$

The following theorem describes invariant harmonic functions on the rays.

Theorem 1.9. *Let u be a positive invariant harmonic function defined in the unit ball $B^n\subset\mathbb{C}^n$ by a positive Borel measure μ on S^{n-1} with the Poisson-Szegö kernel. Let $\zeta\in S^{n-1}$ and $0\le r'\le r<1$.*

if $\alpha > -n$,

$$\left(\frac{1-r}{1-r'}\right)^{n+2\alpha}\left(\frac{1+r'}{1+r}\right)^n u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{n+2\alpha}\left(\frac{1-r'}{1-r}\right)^n u(r'\zeta)$$

If $\alpha < -n$,

$$\left(\frac{1+r}{1+r'}\right)^{-2n-2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{n+2\alpha} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1-r}{1-r'}\right)^{n+2\alpha}\left(\frac{1+r'}{1+r}\right)^n u(r'\zeta)$$

For $r' = 0$, the above becomes

$$\frac{(1-r)^{n+2\alpha}}{(1+r)^n}u(0) \leq u(r\zeta) \leq \frac{(1+r)^{n+2\alpha}}{(1-r)^n}u(0)$$

for $\alpha > -n$, and

$$\frac{(1+r)^{n+2\alpha}}{(1-r)^n}u(0) \leq u(r\zeta) \leq \frac{(1-r)^{n+2\alpha}}{(1+r)^n}u(0)$$

for $\alpha < -n$.

2. Proofs of Theorem 1.1 and its corollaries

We need the following two lemmas for the proof of Theorem 1.1.

Lemma 2.1. *Let $x \in \mathbb{R}^n$, $|x| = r$, $\zeta \in S^{n-1}$.*

If $\lambda > -\frac{n}{2}$ then

$$-\frac{(n+2\lambda-(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \leq \frac{\partial}{\partial r}\frac{(1-r^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} \leq \frac{(n+2\lambda+(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \tag{2.1}$$

If $\lambda < -\frac{n}{2}$, then

$$\frac{(n+2\lambda+(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \leq \frac{\partial}{\partial r}\frac{(1-r^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} \leq -\frac{(n+2\lambda-(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} \tag{2.2}$$

Proof. Write $x = |x|\eta = r\eta$, $\eta \cdot \zeta = \sum_{i=1}^n \eta_i\zeta_i$.

$$\frac{\partial}{\partial r}|x-\zeta|^2 = \frac{\partial}{\partial r}(|x|^2 - 2r\eta\cdot\zeta + 1) = 2(r - \eta\cdot\zeta),$$

then

$$\begin{aligned}\frac{\partial}{\partial r}|x-\zeta|^{n+2\lambda} &= \frac{\partial}{\partial r}(|x-\zeta|^2)^{\frac{n+2\lambda}{2}} \\ &= \frac{n+2\lambda}{2}(|x-\zeta|^2)^{\frac{n+2\lambda}{2}-1}\frac{\partial}{\partial r}|x-\zeta|^2 \\ &= (n+2\lambda)\ |x-\zeta|^{n+2\lambda-2}\ (r-\eta\cdot\zeta),\end{aligned}$$

and

$$\begin{aligned}&\frac{\partial}{\partial r}\frac{(1-r^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} \\ =&\frac{(1+2\lambda)(1-r^2)^{2\lambda}(-2r)|x-\zeta|^{n+2\lambda} - (1-r^2)^{1+2\lambda}\frac{\partial}{\partial r}|x-\zeta|^{n+2\lambda}}{|x-\zeta|^{2(n+2\lambda)}} \\ =&\frac{-2(1+2\lambda)(1-r^2)^{2\lambda}r|x-\zeta|^{n+2\lambda} - (1-r^2)^{1+2\lambda}(n+2\lambda)|x-\zeta|^{n+2\lambda-2}(r-\eta\cdot\zeta)}{|x-\zeta|^{2(n+2\lambda)}} \\ =&\frac{-2(1+2\lambda)(1-r^2)^{2\lambda}r|x-\zeta|^2 - (1-r^2)^{1+2\lambda}(n+2\lambda)(r-\eta\cdot\zeta)}{|x-\zeta|^{n+2\lambda+2}}.\end{aligned} \tag{2.3}$$

To prove the right side inequality in (2.1), it suffices to show

$$-2(1+2\lambda)r|x-\zeta|^2-(1-r^2)(n+2\lambda)(r-\eta\cdot\zeta)\leq(n+2\lambda+(n-2\lambda-2)r)|x-\zeta|^2,$$

which is equivalent to

$$-(n+2\lambda)(1-r^2)(r-\eta\cdot\zeta)\leq(n+2\lambda)(1+r)|x-\zeta|^2.$$

For $\lambda>-\frac{n}{2}$, the above becomes

$$-(1-r^2)(r-\eta\cdot\zeta)\leq(1+r)|x-\zeta|^2,$$

or

$$-(1-r)(r-\eta\cdot\zeta)\leq r^2-2\eta\cdot\zeta+1,$$

which, after a simple simplification, is equivalent to

$$\eta\cdot\zeta\leq 1$$

The inequality is true since $\zeta,\eta\in S^{n-1}$. To prove the left side inequality in (2.1), it suffices to show (using the result of (2.3))

$$-2(1+2\lambda)r|x-\zeta|^2-(1-r^2)(n+2\lambda)(r-\eta\cdot\zeta)\geq-(n+2\lambda-(n-2\lambda-2)r)|x-\zeta|^2,$$

which is equivalent to

$$(n+2\lambda)(1-r^2)(r-\eta\cdot\zeta)\leq(n+2\lambda)(1-r)|x-\zeta|^2.$$

For $\lambda>-\frac{n}{2}$, the inequality is equivalent to

$$(1-r^2)(r-\eta\cdot\zeta)\leq(1-r)|x-\zeta|^2,$$

which is, after a simplification,

$$-\eta\cdot\zeta\leq 1,$$

true since $\zeta,\eta\in S^{n-1}$. The proof of (2.2) for $\lambda<-\frac{n}{2}$ is parallel. This completes the proof of Lemma 2.1. ∎

Lemma 2.2. *Let u be a positive invariant harmonic function in B^n defined by a positive Borel measure on S^{n-1} with the Poisson kernel.*

If $\lambda>-\frac{n}{2}$,

$$-\frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}u(x)\leq\frac{\partial u(x)}{\partial r}\leq\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}u(x). \tag{2.4}$$

If $\lambda<-\frac{n}{2}$,

$$\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}u(x)\leq\frac{\partial u(x)}{\partial r}\leq-\frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}u(x). \tag{2.5}$$

Proof. By the Poisson integral representation of u in B^n,

$$u(x) = \int_{S^{n-1}} \frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta)$$

for a positive Borel measure μ. By (2.1) in Lemma 2.1 and μ being a positive measure,

$$\begin{aligned}\int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\right) d\mu(\zeta) &\le \int_{S^{n-1}} \frac{(n+2\lambda+(n-2\lambda-2)r)(1-r^2)^{2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta)\\ &= \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}\int_{S^{n-1}} \frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}} d\mu(\zeta)\\ &= \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}\, u(x)\end{aligned}$$

when $\lambda > -\frac{n}{2}$. It follows that

$$\frac{\partial u(x)}{\partial r} = \int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|x|^2)^{1+2\lambda}}{|x-\zeta|^{n+2\lambda}}\right) d\mu(\zeta) \le \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2}\, u(x).$$

The left side inequality in (2.4) can be proved in the same manner. For the equality case, consider $u_y(x) = u(x,y) = \dfrac{(1-|x|^2)^{1+2\lambda}}{|x-y|^{n+2\lambda}}$ which is invariant harmonic in $\mathbb{R}^n \setminus \{y\}$ for $y \in S^{n-1}$. A simple calculation shows that the equalities hold for $u_y(x)$ when $x = |x|y$ and $x = -|x|y$ respectively. The proof of (2.5) is similar. This completes the proof of Lemma 2.2. ∎

Now we prove Theorem 1.1.

Proof. Consider $\varphi(r) = \dfrac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}$ and $\psi(r) = \dfrac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}$ for $0 \le r < 1$.

$$\frac{\varphi'}{\varphi} = -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2},$$

$$\frac{\psi'}{\psi} = \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2}.$$

Given $\omega \in S^{n-1}$, consider

$$\begin{aligned} I(r,\omega) &= \varphi(r)u(r\omega),\\ J(r,\omega) &= \psi(r)u(r\omega).\end{aligned}$$

To show Theorem 1.1, it suffices to show that $I(r,\omega)$ is decreasing (increasing) and $J(r,\omega)$ is increasing (decreasing) in r for $0 \le r < 1$ when $\lambda > -\frac{n}{2}$ (when

$\lambda < -\frac{n}{2}$). By (2.4) in Lemma 2.2, for $\lambda > -\frac{n}{2}$,

$$\begin{aligned}\frac{d}{dr}(\log I(r,\omega)) &= \frac{\varphi'}{\varphi} + \frac{u_r'}{u} \\ &= -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} + \frac{u_r'}{u} \\ &\le -\frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} + \frac{(n+2\lambda+(n-2\lambda-2)r)}{1-r^2} \\ &= 0.\end{aligned}$$

Therefore $\log I(r,\omega)$ is decreasing in r, and so is $I(r,\omega)$. Similarly,

$$\begin{aligned}\frac{d}{dr}(\log J(r,\omega)) &= \frac{\psi'}{\psi} + \frac{u_r'}{u} \\ &= \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} + \frac{u_r'}{u} \\ &\ge \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} - \frac{(n+2\lambda-(n-2\lambda-2)r)}{1-r^2} \\ &= 0.\end{aligned}$$

Hence, $J(r,\omega)$ is increasing in r. For $\lambda > -\frac{n}{2}$ and $y \in S^{n-1}$,

$$\begin{aligned}\lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,y) &= \lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\frac{(1-|r|^2)^{1+2\lambda}}{|r\zeta-y|^{n+2\lambda}} \\ &= \lim_{r\to 1}\frac{(1-r)^{n+2\lambda}}{|r\zeta-y|^{n+2\lambda}} \searrow \delta(\zeta,y) = \begin{cases}1, & \zeta = y\\ 0, & \zeta \ne y\end{cases}\end{aligned}$$

by applying the monotonicity properties in Theorem 1.1 to $u = P_\lambda$. By Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{r\to 1}(1-r)^{n-1}u(r\zeta) &= \lim_{r\to 1}(1-r)^{n-1}\int_{S^{n-1}}P_\lambda(r\zeta,\xi)d\mu(\xi) \\ &= \lim_{r\to 1}(1+r)^{1+2\lambda}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi) \\ &= 2^{1+2\lambda}\mu(\{\zeta\}).\end{aligned}$$

Similarly, $\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,y)=\frac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}$ is increasing in r for $\lambda>-\frac{n}{2}$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}\lim_{r\to1}\frac{u(r\zeta)}{(1-r)^{1+2\lambda}} &= \lim_{r\to1}\frac{1}{(1-r)^{1+2\lambda}}\int_{S^{n-1}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to1}\frac{1}{(1+r)^{n-1}}\int_{S^{n-1}}\lim_{r\to1}\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \frac{1}{2^{n-1}}\int_{S^{n-1}}\lim_{r\to1}\frac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}d\mu(\xi)\\ &= \int_{S^{n-1}}\frac{2^{1+2\lambda}}{|\zeta-\xi|^{n+2\lambda}}d\mu(\xi).\end{aligned}$$

For $\lambda<-\frac{n}{2}$, the monotonicity of $I(r,\omega)$ and $J(r,\omega)$ is proved similarly to the case $\lambda>-\frac{n}{2}$ using (2.5) instead of (2.4) in Lemma 2.2.

$$\lim_{r\to1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,y)=\lim_{r\to1}\frac{|r\zeta-y|^{-(n+2\lambda)}}{(1-r)^{-(n+2\lambda)}}\nearrow\begin{cases}1, & \zeta=y\\ \infty, & \zeta\neq y\end{cases}$$

when $\lambda<-\frac{n}{2}$. Therefore,

$$\begin{aligned}\lim_{r\to1}(1-r)^{n-1}u(r\zeta) &= \lim_{r\to1}(1+r)^{1+2\lambda}\int_{S^{n-1}}\lim_{r\to1}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \begin{cases}2^{1+2\lambda}\mu(\{\zeta\}), & if\ \mu(\{\zeta\}^c)=0;\\ \infty, & if\ \mu(\{\zeta\}^c)>0.\end{cases}\end{aligned}$$

Similarly, $\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,y)=\frac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}$ is decreasing in r for $\lambda<-\frac{n}{2}$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}\lim_{r\to1}\frac{u(r\zeta)}{(1-r)^{1+2\lambda}} &= \lim_{r\to1}\frac{1}{(1-r)^{1+2\lambda}}\int_{S^{n-1}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to1}\frac{1}{(1+r)^{n-1}}\int_{S_{n-1}}\lim_{r\to1}\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= \frac{1}{2^{n-1}}\int_{S^{n-1}}\lim_{r\to1}\frac{(1+r)^{n+2\lambda}}{|r\zeta-\xi|^{n+2\lambda}}d\mu(\xi)\\ &= \int_{S^{n-1}}\frac{2^{1+2\lambda}}{|\zeta-\xi|^{n+2\lambda}}d\mu(\xi)\end{aligned}$$

This completes the proof of Theorem 1.1. ∎

The proof of Corollary 1.2 is straightforward and is omitted. The proof of Corollary 1.3 follows.

Proof.

$$\begin{aligned}(1-r)^{n+2\lambda}U(r\zeta) &= \int_{S^{n-1}} \frac{(1-r)^{n+2\lambda}}{|r\zeta-\eta|^{n+2\lambda}}d\mu(\eta)\\ &= \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\int_{S^{n-1}} \frac{(1-r^2)^{1+2\lambda}}{|r\zeta-\eta|^{n+2\lambda}}d\mu(\eta)\\ &= \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}u(r\zeta)\end{aligned}$$

which is decreasing (increasing) in r for $\lambda > -\frac{n}{2}$ ($\lambda < -\frac{n}{2}$) by Theorem 1.1. This completes the proof of Corollary 1.3. ∎

Corollaries 1.4 and 1.5 are special cases of Theorem 1.1. Corollary 1.6 is a straightforward generalization from B^n to $B^n(R)$. The following is the proof of Corollary 1.7.

Proof. $0 \le r' \le r < 1$. By the maximum principle, there is $\zeta \in S^{n-1}$ such that $u(r\zeta) = \max_{|x|=r} u(x)$.

If $\lambda > -\frac{n}{2}$, Theorem 1.1 implies

$$\begin{aligned}\frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\max_{|x|=r} u(x) &= \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}u(r\zeta)\\ &\le \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}}u(r'\zeta) \le \frac{(1-r')^{n-1}}{(1+r')^{1+2\lambda}}\max_{|x|=r'} u(x)\end{aligned}$$

Similarly, there is $\xi \in S^{n-1}$ such that $u(r\xi) = \min_{|x|=r} u(x)$. When $\lambda > -\frac{n}{2}$, Theorem 1.1 yields

$$\begin{aligned}\frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}\min_{|x|=r} u(x) &= \frac{(1+r)^{n-1}}{(1-r)^{1+2\lambda}}u(r\xi)\\ &\ge \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}}u(r'\xi) \ge \frac{(1+r')^{n-1}}{(1-r')^{1+2\lambda}}\min_{|x|=r'} u(x)\end{aligned}$$

The proof for $\lambda < -\frac{n}{2}$ is parallel. This completes the proof of Corollary 1.7. ∎

3. Proof of Theorem 1.8

In the following, B^n denotes the unit ball in $\mathbb{C}^n$ and $S^{n-1} = \partial B^n$ the sphere. We need the following three lemmas for the proof of Theorem 1.8.

Lemma 3.1. *If* $a \in \mathbb{C}, |a| \le 1$, *then for* $0 \le r \le 1$,

$$1 + r|a|^2 \ge (1+r)\mathrm{Re}(a) \tag{3.1}$$

$$1 - r|a|^2 \ge (-1+r)\mathrm{Re}(a) \tag{3.2}$$

Proof. $|a| \le 1$, so $-1 \le -|a| \le \mathrm{Re}(a) \le |a| \le 1$ and $\mathrm{Re}(a)^2 \le |a|^2$.

If $|a|^2 \ge \mathrm{Re}(a)$, then $1 + r|a|^2 \ge \mathrm{Re}(a) + r\mathrm{Re}(a)$ so (3.1) holds.

If $|a|^2 < \mathrm{Re}(a)$, consider $f(r) = 1+r|a|^2-(1+r)\mathrm{Re}(a)$, $f'(r) = |a|^2-\mathrm{Re}(a) < 0$. So $f(r)$ decreases in $r \in [0,1]$. $f(1) = 1 + |a|^2 - 2\mathrm{Re}(a) > 1 + \mathrm{Re}(a)^2 - 2\mathrm{Re}(a) = (1 - Re(a))^2 \ge 0$. So $f(r) \ge 0$ and (3.1) holds.

For the second inequality, $1+\mathrm{Re}(a) \ge |a|^2+\mathrm{Re}(a) \ge r|a|^2+r\mathrm{Re}(a)$, so $1-r|a|^2 \ge (-1 + r)\mathrm{Re}(a)$ and (3.2) holds. ∎

Lemma 3.2. *Let* $z \in B^n$, $|z| = r$, $\zeta \in S^{n-1}$.

If $\alpha > -n$ *then*

$$-\frac{(2n+2\alpha + 2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \le \frac{\partial}{\partial r}\frac{(1-r^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} \le \frac{(2n+2\alpha-2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \tag{3.3}$$

If $\alpha < -n$, *then*

$$\frac{(2n+2\alpha-2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \le \frac{\partial}{\partial r}\frac{(1-r^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} \le -\frac{(2n+2\alpha+2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}} \tag{3.4}$$

Proof. Let $z = |z|\eta = r\eta$.

$$\frac{\partial}{\partial r}|1-z\cdot\overline{\zeta}|^2 = \frac{\partial}{\partial r}(1 - 2r\mathrm{Re}(\eta\cdot\overline{\zeta}) + r^2|\eta\cdot\overline{\zeta}|^2) = 2(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})),$$

and

$$\begin{aligned}\frac{\partial}{\partial r}|1-z\cdot\overline{\zeta}|^{2n+2\alpha} &= \frac{\partial}{\partial r}(|1-z\cdot\overline{\zeta}|^2)^{n+\alpha}\\ &= (n+\alpha)|(1-z\cdot\overline{\zeta}|^2)^{n+\alpha-1}\frac{\partial}{\partial r}|1-z\cdot\overline{\zeta}|^2\\ &= 2(n+\alpha)|1-z\cdot\overline{\zeta}|^{2n+2\alpha-2}(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta})).\end{aligned}$$

We have

$$\begin{aligned}\frac{\partial}{\partial r}\frac{(1-r^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}} &= \frac{(n+2\alpha)(1-r^2)^{n+2\alpha-1}(-2r)|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{4n+4\alpha}}\\ &\quad - \frac{(1-r^2)^{n+2\alpha}\frac{\partial}{\partial r}|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{4n+4\alpha}}\\ &= \frac{-2(n+2\alpha)(1-r^2)^{n+2\alpha-1}r|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{4n+4\alpha}}\\ &\quad - \frac{(1-r^2)^{n+2\alpha}2(n+\alpha)|1-z\cdot\overline{\zeta}|^{2n+2\alpha-2}(r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta}))}{|1-z\cdot\overline{\zeta}|^{4n+4\alpha}}\\ &= \frac{-2(n+2\alpha)(1-r^2)^{n+2\alpha-1}r|1-z\cdot\overline{\zeta}|^2}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha+2}}\\ &\quad - \frac{(1-r^2)^{n+2\alpha}2(n+\alpha)(r|\eta\cdot\overline{\zeta}|^2 - \mathrm{Re}(\eta\cdot\overline{\zeta}))}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha+2}}\end{aligned} \tag{3.5}$$

To prove the right side inequality of (3.3), it suffices to prove

$$-2(n+2\alpha)r|1-z\cdot\overline{\zeta}|^2-(1-r^2)(2n+2\alpha)(r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta})) \le (2n+2\alpha-2\alpha r)|1-z\cdot\overline{\zeta}|^2$$

which is equivalent to

$$-(1-r^2)2(n+\alpha)(r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta})) \le 2(n+\alpha)(1+r)|1-z\cdot\overline{\zeta}|^2.$$

For $\alpha > -n$, the above inequality is equivalent to

$$-(1-r)((r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta})) \le |1-z\cdot\overline{\zeta}|^2$$

which is, after a simple simplification,

$$(1+r)\mathrm{Re}(\eta\cdot\overline{\zeta}) \le 1+r|\eta\cdot\overline{\zeta}|^2.$$

The inequality is true by (3.1) in Lemma 3.1. To prove the left side inequality of (3.3), it suffices to show

$$-2(n+2\alpha)r|1-z\cdot\overline{\zeta}|^2-(1-r^2)(2n+2\alpha)(r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta})) \ge -(2n+2\alpha+2\alpha r)|1-z\cdot\overline{\zeta}|^2$$

which is equivalent to

$$-(1-r^2)2(n+\alpha)(r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta})) \ge -2(n+\alpha)(1-r)|1-z\cdot\overline{\zeta}|^2.$$

For $\alpha > -n$, the above inequality becomes

$$(1+r)((r|\eta\cdot\overline{\zeta}|^2-\mathrm{Re}(\eta\cdot\overline{\zeta})) \le |1-z\cdot\overline{\zeta}|^2$$

which is, after a simple simplification,

$$(-1+r)\mathrm{Re}(\eta\cdot\overline{\zeta}) \le 1-r|\eta\cdot\overline{\zeta}|^2.$$

The inequality is true by (3.2) in Lemma 3.1. The proof of (3.4) is parallel to that of (3.3), using the same inequalities in Lemma 3.1. This completes the proof of Lemma 3.2. ∎

Lemma 3.3. *Let $u(z)$ be a positive invariant harmonic function in B^n defined by a positive Borel measure on S^{n-1} with the Poisson-Szegö kernel, $|z| = r$.*
If $\alpha > -n$,

$$-\frac{(2n+2\alpha+2\alpha r)}{1-r^2}u(z) \le \frac{\partial u(z)}{\partial r} \le \frac{(2n+2\alpha-2\alpha r)}{1-r^2}u(z). \tag{3.6}$$

If $\alpha < -n$,

$$\frac{(2n+2\alpha-2\alpha r)}{1-r^2}u(z) \le \frac{\partial u(z)}{\partial r} \le -\frac{(2n+2\alpha+2\alpha r)}{1-r^2}u(z). \tag{3.7}$$

Proof. By the Poisson-Szegö integral representation of u in B^n,

$$u(z) = \int_{S^{n-1}} \frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}d\mu(\zeta)$$

for a positive Borel measure μ on S^{n-1}. By (3.3) in Lemma 3.2 and μ being a positive measure,

$$\begin{aligned}\int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}\right)d\mu(\zeta) &\le \int_{S^{n-1}} \frac{(n+2\alpha-2\alpha r)(1-r^2)^{n+2\alpha-1}}{|1-z\cdot\overline{\zeta}|^{n+2\alpha}}d\mu(\zeta)\\ &= \frac{(n+2\alpha-2\alpha r)}{1-r^2}\int_{S^{n-1}} \frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}d\mu(\zeta)\\ &= \frac{(n+2\alpha-2\alpha r)}{1-r^2}u(z)\end{aligned}$$

when $\alpha > -n$. It follows that

$$\frac{\partial u(z)}{\partial r} = \int_{S^{n-1}} \frac{\partial}{\partial r}\left(\frac{(1-|z|^2)^{n+2\alpha}}{|1-z\cdot\overline{\zeta}|^{2n+2\alpha}}\right)d\mu(\zeta) \le \frac{(n+2\alpha-2\alpha r)}{1-r^2}u(z).$$

The left side inequality in (3.6) is proved similarly. For the equality case, consider $u_w(z) = P_\alpha(z,w) = \dfrac{(1-|z|^2)^{n+2\alpha}}{|z-w|^{2n+2\alpha}}$. It is known that $u_w(z)$ is invariant harmonic in $\mathbb{C}^n \setminus \{w\}$ for $w \in S^{n-1}$. A simple calculation shows that the equalities in (3.6) hold for $u_w(z)$ when $z = |z|w$ and $z = -|z|w$ respectively. The proof of (3.7) is parallel to that of (3.6), using (3.4) instead of (3.3) in Lemma 3.2. This completes the proof of Lemma 3.3. ∎

Now we prove Theorem 1.8.

Proof. Consider $\varphi(r) = \dfrac{(1-r)^n}{(1+r)^{n+2\alpha}}$, $\psi(r) = \dfrac{(1+r)^n}{(1-r)^{n+2\alpha}}$ for $0 \le r < 1$.

$$\begin{aligned}\frac{\varphi'}{\varphi} &= -\frac{2n+2\alpha-2\alpha r}{1-r^2},\\ \frac{\psi'}{\psi} &= \frac{2n+2\alpha+2\alpha r}{1-r^2}.\end{aligned}$$

Given $\omega \in S^{n-1}$, consider

$$\begin{aligned}I(r,\omega) &= \varphi(r)u(r\omega),\\ J(r,\omega) &= \psi(r)u(r\omega).\end{aligned}$$

To show Theorem 1.8, it suffices to show that $I(r,\omega)$ is decreasing (increasing) and $J(r,\omega)$ is increasing (decreasing) in r when $\alpha > -n$ (when $\alpha < -n$). By (3.6) in Lemma 3.3, when $\alpha > -n$,

$$\begin{aligned}\frac{\partial}{\partial r}\log I(r,\omega) &= \frac{\varphi'}{\varphi} + \frac{u'_r}{u}\\ &= -\frac{2n+2\alpha-2\alpha r}{1-r^2} + \frac{u'_r}{u}\\ &\le -\frac{2n+2\alpha-2\alpha r}{1-r^2} + \frac{2n+2\alpha-2\alpha r}{1-r^2}\\ &= 0.\end{aligned}$$

Therefore $\log I(r,\omega)$ is decreasing in r, and so is $I(r,\omega)$. Similarly,

$$\begin{aligned}\frac{\partial}{\partial r}\log J(r,\omega) &= \frac{\psi'}{\psi}+\frac{u_r'}{u}\\ &= \frac{2n+2\alpha+2\alpha r}{1-r^2}+\frac{u_r'}{u}\\ &\geq \frac{2n+2\alpha+2\alpha r}{1-r^2}-\frac{2n+2\alpha+2\alpha r}{1-r^2}\\ &= 0.\end{aligned}$$

Hence, $J(r,\omega)$ is increasing in r. For $\alpha>-n$ and $\zeta, w\in S^{n-1}$,

$$\begin{aligned}\lim_{r\to1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}P_\alpha(r\zeta,w) &= \lim_{r\to1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}\frac{(1-|r|^2)^{n+2\alpha}}{|1-r\zeta\cdot\overline{w}|^{2n+2\alpha}}\\ &= \lim_{r\to1}\frac{(1-r)^{2n+2\alpha}}{|1-r\zeta\cdot\overline{w}|^{2n+2\alpha}}\searrow\delta(\zeta,w)\end{aligned}$$

by applying the monotonicity results in Theorem 1.8 to $u=P_\alpha$. By Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{r\to1}(1-r)^n u(r\zeta) &= \lim_{r\to1}(1-r)^n\int_{S^{n-1}}P_\alpha(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to1}(1+r)^{n+2\alpha}\int_{S^{n-1}}\lim_{r\to1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}P_\lambda(r\zeta,\xi)d\mu(\xi)\\ &= 2^{n+2\alpha}\mu(\{\zeta\}).\end{aligned}$$

Similarly, $\dfrac{(1+r)^n}{(1-r)^{n+2\alpha}}P_\alpha(r\zeta,w)=\dfrac{(1+r)^{2n+2\alpha}}{|1-r\zeta\cdot\overline{w}|^{2n+2\alpha}}$ is increasing in r for $\alpha>-n$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}\lim_{r\to1}\frac{u(r\zeta)}{(1-r)^{n+2\alpha}} &= \lim_{r\to1}\frac{1}{(1-r)^{n+2\alpha}}\int_{S^{n-1}}P_\alpha(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to1}\frac{1}{(1+r)^n}\int_{S^{n-1}}\lim_{r\to1}\frac{(1+r)^n}{(1-r)^{n+2\alpha}}P_\alpha(r\zeta,\xi)d\mu(\xi)\\ &= \frac{1}{2^n}\int_{S^{n-1}}\lim_{r\to1}\frac{(1+r)^{2n+2\alpha}}{|1-r\zeta\cdot\bar{\xi}|^{2n+2\alpha}}d\mu(\xi)\\ &= \int_{S^{n-1}}\frac{2^{n+2\alpha}}{|\zeta-\xi|^{2n+2\alpha}}d\mu(\xi).\end{aligned}$$

For $\alpha<-n$, the monotonicity of $I(r,\omega)$ and $J(r,\omega)$ is proved similarly by applying (3.7) in Lemma 3.3.

$$\lim_{r\to1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}P_\alpha(r\zeta,w)=\lim_{r\to1}\frac{|1-r\zeta\cdot\overline{w}|^{-(2n+2\alpha)}}{(1-r)^{-(2n+2\alpha)}}\nearrow\begin{cases}1, & \zeta=w\\ \infty, & \zeta\neq w\end{cases}$$

when $\alpha < -n$. Therefore,

$$\begin{aligned}\lim_{r\to 1}(1-r)^n u(r\zeta) &= \lim_{r\to 1}(1+r)^{n+2\alpha}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1-r)^n}{(1+r)^{n+2\alpha}}P_\alpha(r\zeta,\xi)d\mu(\xi)\\ &= \begin{cases}2^{n+2\alpha}\mu(\{\zeta\}), & if\ \mu(\{\zeta\}^c)=0;\\ \infty, & if\ \mu(\{\zeta\}^c)>0.\end{cases}\end{aligned}$$

Similarly, $\frac{(1+r)^n}{(1-r)^{n+2\alpha}}P_\alpha(r\zeta,w) = \frac{(1+r)^{2n+2\alpha}}{|1-r\zeta\cdot\overline{w}|^{2n+2\alpha}}$ is decreasing in r for $\alpha < -n$. By Lebesgue's monotone convergence theorem,

$$\begin{aligned}\lim_{r\to 1}\frac{u(r\zeta)}{(1-r)^{n+2\alpha}} &= \lim_{r\to 1}\frac{1}{(1-r)^{n+2\alpha}}\int_{S^{n-1}}P_\alpha(r\zeta,\xi)d\mu(\xi)\\ &= \lim_{r\to 1}\frac{1}{(1+r)^n}\int_{S_{n-1}}\lim_{r\to 1}\frac{(1+r)^n}{(1-r)^{n+2\alpha}}P_\alpha(r\zeta,\xi)d\mu(\xi)\\ &= \frac{1}{2^n}\int_{S^{n-1}}\lim_{r\to 1}\frac{(1+r)^{2n+2\alpha}}{|1-r\zeta\cdot\overline{\xi}|^{2n+2\alpha}}d\mu(\xi)\\ &= \int_{S^{n-1}}\frac{2^{n+2\alpha}}{|\zeta-\xi|^{2n+2\alpha}}d\mu(\xi)\end{aligned}$$

This completes the proof of Theorem 1.8. ∎

Remark. Notice that the monotonicity of the auxiliary functions φ and ψ in the proofs of Theorem 1.1 and Theorem 1.8 may vary depending on the values of the parameter λ (or α) and the dimension n. When $\lambda > -\frac{n}{2}$ (or $\alpha > -n$), we have $\varphi' < 0$ and $\psi' > 0$, i.e. φ increases and ψ decreases in r for $0 < r < 1$. For $\lambda < -\frac{n}{2}$ (or $\alpha < -n$), the monotonicity does not necessarily hold. For example, in the real case in Theorem 1.1, for $\lambda < -\frac{n}{2}$,

$$\varphi(r) = \frac{(1-r)^{n-1}}{(1+r)^{1+2\lambda}}\ , \qquad \varphi'(r)\begin{cases}>0, & r\in\left(0,\dfrac{-2\lambda-n}{-2\lambda+(n-2)}\right)\\ <0, & r\in\left(\dfrac{-2\lambda-n}{-2\lambda+(n-2)},1\right)\end{cases}$$

i.e. the monotonicity may change for certain combinations of n and λ. However, the monotonicity of $\varphi(r)u(r\zeta)$ and $\psi(r)u(r\zeta)$ holds.

4. Proofs of Theorem 1.2 and Theorem 1.9

The proofs for the two theorems are based on the following lemma.

Lemma 4.1. *Let $f(r)$ be a positive function on $r \in [0,1)$. If for $a, b \in \mathbb{R}$,*

$$-\frac{a+br}{1-r^2}f(r) \le f'(r) \le \frac{a-br}{1-r^2}f(r), \tag{4.1}$$

then for $0 \le r' \le r < 1$,

$$\left(\frac{1+r}{1+r'}\right)^{-a}\left(\frac{1-r^2}{1-r'^2}\right)^{\frac{b+a}{2}} f(r') \le f(r) \le \left(\frac{1+r}{1+r'}\right)^{a}\left(\frac{1-r^2}{1-r'^2}\right)^{\frac{b-a}{2}} f(r'). \tag{4.2}$$

Proof.

$$\int \frac{a-br}{1-r^2}dr = a\ln(1+r) + \frac{1}{2}(b-a)\ln(1-r^2) + C$$

Thus for $0 \le r' \le r'' < 1$, by (4.1),

$$\ln f(r'') - \ln f(r') = \int_{r'}^{r''} \frac{f'(r)}{f(r)}dr \le \int_{r'}^{r''} \frac{a-br}{1-r^2}dr \le \ln\left(\frac{1+r''}{1+r'}\right)^{a}\left(\frac{1-r''^2}{1-r'^2}\right)^{\frac{b-a}{2}}$$

i.e. the right side inequality in (4.2) holds. Similarly, by the left side of (4.1),

$$\ln f(r'') - \ln f(r') \ge \int_{r'}^{r''} -\frac{a+br}{1-r^2}dr \ge \ln\left(\frac{1+r''}{1+r'}\right)^{-a}\left(\frac{1-r''^2}{1-r'^2}\right)^{\frac{b+a}{2}}.$$

i.e. the left side inequality in (4.2) holds. ∎

Now we prove Theorem 1.2.

Proof. If $\lambda > -\frac{n}{2}$, $u(r\zeta)$ satisfies (2.4) in Lemma 2.2. Therefore (4.1) holds with $f(r) = u(r\zeta),\ a = n+2\lambda,\ b = -n+2\lambda+2$. Let $0 \le r' \le r < 1$. (4.2) in Lemma 4.1 implies

$$\left(\frac{1+r}{1+r'}\right)^{-n-2\lambda}\left(\frac{1-r^2}{1-r'^2}\right)^{2\lambda+1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{n+2\lambda}\left(\frac{1-r^2}{1-r'^2}\right)^{-n+1} u(r'\zeta).$$

If $\lambda < -\frac{n}{2}$, $u(r\zeta)$ satisfies (2.5) in Lemma 2.2, thus (4.1) holds with $f(r) = u(r\zeta),\ a = -n-2\lambda,\ b = -n+2\lambda+2$. Applying (4.2),

$$\left(\frac{1+r}{1+r'}\right)^{n+2\lambda}\left(\frac{1-r^2}{1-r'^2}\right)^{-n+1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{-n-2\lambda}\left(\frac{1-r^2}{1-r'^2}\right)^{2\lambda+1} u(r'\zeta).$$

This completes the proof of Theorem 1.2. ∎

The proof of Theorem 1.9 is similar to that of Theorem 1.2.

Proof. If $\alpha > -n$, $u(r\zeta)$ satisfies (3.6) in Lemma 3.3. Therefore (4.1) holds with $f(r) = u(r\zeta),\ a = 2n+2\alpha,\ b = 2\alpha$. Let $0 \le r' \le r < 1$. (4.2) in Lemma 4.1 implies

$$\left(\frac{1+r}{1+r'}\right)^{-2n-2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{n+2\alpha} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2n+2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{-n} u(r'\zeta).$$

If $\alpha < -n$, $u(r\zeta)$ satisfies (3.7) in Lemma 3.3, thus (4.1) holds with $f(r) = u(r\zeta),\ a = -2n - 2\alpha,\ b = 2\alpha$. From (4.2),

$$\left(\frac{1+r}{1+r'}\right)^{2n+2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{-n} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{-2n-2\alpha}\left(\frac{1-r^2}{1-r'^2}\right)^{n+2\alpha} u(r'\zeta).$$

This completes the proof of Theorem 1.9. ∎

Most results in this paper are on the function values at two points in B^n on the same ray. Similar results can be obtained for any two points in B^n ([5]).

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