A REMARK ON FOURIER SERIES OF A COMPLEX MEASURE

YIFEI PAN

1. Introduction

Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus. The space of complex measure on $\mathbb{T}$ will be denoted by $\mathcal{M}(\mathbb{T})$. Any $\mu \in \mathcal{M}(\mathbb{T})$ has associated with it a Fourier series

$$\mu \sim \sum_{n=-\infty}^{\infty} \hat{\mu}(n) e^{-2\pi i n x}$$

where

$$\hat{\mu}(n) = \int_0^1 e^{-2\pi i n x} d\mu(x).$$

The symmetric partial sum is defined as follows

$$S_N(\mu)(x) = \sum_{n=-N}^{N} \hat{\mu}(n) e^{2\pi i n x}.$$

The Cesaro mean is defined to be

$$\sigma_N(\mu)(x) = \frac{1}{N} \sum_{n=0}^{N-1} S_n(\mu)(x).$$

In this note we prove some basic results about Fourier series of a complex measure. It seems that they are not covered in any textbooks on Fourier series.

Theorem 1.1. *Let μ be a complex measure on $\mathbb{T}$. Then for each $x \in \mathbb{T}$*

$$S_N(\mu)(x) \sim 2\mu(\{x\})N,$$
$$\sigma_N(\mu)(x) \sim \mu(\{x\})N$$

as $N \to \infty$.

The following gives a formula for point mass of a complex measure.

Corollary 1.2. *Let μ be a complex measure on $\mathbb{T}$. Then for each $x \in \mathbb{T}$*

$$\mu(\{x\}) = \lim_{N\to\infty} \frac{S_N(\mu)(x)}{2N} = \lim_{N\to\infty} \frac{\sigma_N(\mu)(x)}{N}.$$

In particular, if $\mu(\{x\}) \neq 0$ for some x, then the Fourier series is divergent at x.

It is well-known that there are at most countably many points for which $\mu(\{x\}) \neq 0$. We have

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Corollary 1.3. *Let μ be a complex measure on $\mathbb{T}$. Then except at most countably many points on $\mathbb{T}$, one has*

$$\lim_{N\to\infty}\frac{S_N(\mu)(x)}{N}=\lim_{N\to\infty}\frac{\sigma_N(\mu)(x)}{N}=0.$$

We remark if $d\mu = fdx$ where $f\in L^1(d\mu)$, The corollary follows immediately from the Riemann-Lebesgue lemma, since in this case $\hat{\mu}(n)\to 0$ as $n\to\infty$ already! Now we give the Fourier series on n-dimensional torus $\mathbb{T}^n$. Let μ be a complex measure on $\mathbb{T}^n$. Define, for $\kappa=(\kappa_1,...,\kappa_n)\in\mathbb{Z}^n$

$$\hat{\mu}(\kappa)=\int_{\mathbb{T}^n}e^{-2\pi i\kappa\cdot x}d\mu(x).$$

The "cubical partial sums" ([1])

$$S_N^c(\mu)(x)=\sum_{||\kappa||\le m}\hat{\mu}(\kappa)e^{2\pi i\kappa\cdot x}$$

where $||\kappa||\le m=\max(|\kappa_1|,...,|\kappa_n|)$.

Theorem 1.4. *Let μ be a complex measure on $\mathbb{T}^n$. Then for each $x\in\mathbb{T}^n$, one has*

$$S_N^c(\mu)(x)\sim\mu(\{x\})(2N)^n$$

as $N\to\infty$.

Corollary 1.5. *Let μ be a complex measure on $\mathbb{T}^n$. Then for each $x\in\mathbb{T}^n$*

$$\mu(\{x\})=\lim_{N\to\infty}\frac{S_N^c(\mu)(x)}{(2N)^n}.$$

Now let us give a version of Fourier transforms. Let μ be a complex measure so that $|\mu|(\mathbb{R})<\infty$. Define for $\xi\in\mathbb{R}$

$$\hat{\mu}(\xi)=\int e^{-2\pi i\xi x}d\mu(x).$$

We have the following

Theorem 1.6. *Let μ be a complex measure on $\mathbb{R}$ so that $|\mu|(\mathbb{R})<\infty$. Then*

$$\mu(\{x\})=\lim_{R\to\infty}\frac{1}{2R}\int_{-R}^{R}\hat{\mu}(\xi)e^{2\pi i\xi x}d\xi.$$

Below is the version for $\mathbb{R}^n$. Let μ be a complex measure so that $|\mu|(\mathbb{R}^n)<\infty$. Define for $\xi\in\mathbb{R}^n$

$$\hat{\mu}(\xi)=\int e^{-2\pi i\xi\cdot x}d\mu(x).$$

Theorem 1.7. *Let μ be a complex measure on $\mathbb{R}^n$ so that $|\mu|(\mathbb{R}^n)<\infty$. Then*

$$\mu(\{x\})=\lim_{R\to\infty}\frac{1}{(2R)^n}\int_{-R}^{R}\cdots\int_{-R}^{R}\hat{\mu}(\xi)e^{2\pi i\xi\cdot x}d\xi$$

2. PROOF

Here i s a proof of Theorem 1.1.

Proof. The proof is standard. we have

$$\begin{aligned} S_N(\mu)(x) &= \sum_{n=-N}^{N} \hat{\mu}(n) e^{2\pi i n x} \\ &= \sum_{n=-N}^{N} \int_0^1 e^{-2\pi i n y} d\mu(y) e^{2\pi i n x} \\ &= \sum_{n=-N}^{N} \int_0^1 e^{2\pi i n(x-y)}) d\mu(y) \\ &= \int_0^1 D_N(x-y) d\mu(y), \end{aligned}$$

where $D_N(x) = \sum\limits_{n=-N}^{N} e^{2\pi i n x}$ is the Dirichlet kernel and is given as

$$D_N(x) = \frac{\sin((2N+1)\pi x)}{\sin(\pi x)}.$$

Define

$$\hat{D}_N(x) = \begin{cases} D_N(x) & x \neq 0, 1 \\ 2N+1 & x = 0, 1. \end{cases}$$

Then it is easy to see that $\hat{D}_N(x)$ is continuous on $[0, 1]$. It is also true that there exists a positive constant such that

$$|\hat{D}_N(x)| \leq CN.$$

and We also observe

$$\frac{\hat{D}_N(x-y)}{N} \to 2\delta_{\{x=y\}}$$

as $N \to \infty$. we have

$$\frac{S_N(\mu)(x)}{N} = \int_0^1 \frac{1}{N} \hat{D}_N(x-y) d\mu(y).$$

Applying the dominated convergence theorem, we get

$$\lim_{N\to\infty} \frac{S_N(\mu)(x)}{N} = \int_0^1 \lim_{N\to\infty} \frac{1}{N} \hat{D}_N(x-y) d\mu(y) = \int_0^1 2\delta_{\{x=y\}} d\mu(y) = \mu(\{x\}).$$

Now we prove the second result. It is well-known ([1])

$$\sigma_N(\mu)(x) = \int_0^1 K_N(x-y) d\mu(y)$$

where

$$K_N(x) = \frac{1}{N}\sum_{n=0}^{N-1} D_N(x) = \frac{1}{N}\left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2.$$

Thus the same argument as above applies. ∎

Here is a proof of Theorem 1.3.

Proof.

$$\begin{aligned}
S_N^c(\mu)(x) &= \sum_{||\kappa||\le N} \hat{\mu}(\kappa)e^{2\pi i\kappa\cdot x} \\
&= \sum_{|\kappa_1|\le N} \hat{\mu}(\kappa)e^{2\pi i\kappa\cdot x} \\
&= \sum_{||\kappa||\le N} \int_{\mathbb{T}^n} e^{-2\pi i\kappa\cdot y} d\mu(y) e^{2\pi i\kappa\cdot x} \\
&= \sum_{||\kappa||\le N} \int_{\mathbb{T}^n} e^{2\pi i\kappa\cdot(x-y)} d\mu(y) \\
&= \int_{\mathbb{T}^n} \sum_{||\kappa||\le N} e^{2\pi i\kappa\cdot(x-y)} d\mu(y) \\
&= \int_{\mathbb{T}^n} \prod_{j=1}^{n} D_N(x_j - y_j) d\mu(y),
\end{aligned}$$

where $D_N(x)$ is the Dirichlet kernel. The same arguments as in the one-dimensional case now apply. ∎

Here is a proof of Theorem 1.5.

Proof. By Fubini's theorem,

$$\begin{aligned}
\frac{1}{2R}\int_{-R}^{R} \hat{\mu}(\xi)e^{2\pi i\xi x} d\xi &= \frac{1}{2R}\int_{-R}^{R}\int_{-\infty}^{\infty} e^{-2\pi i\xi y} d\mu(y) e^{2\pi i\xi x} d\xi \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{2R}\int_{-R}^{R} e^{2\pi i\xi(x-y)} d\xi\right) d\mu(y).
\end{aligned}$$

Consider the function

$$F(R,x,y) = \frac{1}{2R}\int_{-R}^{R} e^{2\pi i\xi(x-y)} d\xi.$$

We see $|F(R,x,y)| \leq 1$ and $F(R,x,y) \to 0$ as $R \to \infty$ if $x \neq y$, and $F(R,x,y) = 1$ if $x = y$. So applying the dominated convergence theorem, we have

$$\begin{aligned}\lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} \hat{\mu}(\xi) e^{2\pi i \xi x} d\xi &= \int_{-\infty}^{\infty} \lim_{R\to\infty} \left(\frac{1}{2R} \int_{-R}^{R} e^{2\pi i \xi (x-y)} d\xi \right) d\mu(y) \\ &= \int_{-\infty}^{\infty} \delta_{\{x=y\}} d\mu(y) \\ &= \mu(\{x\}).\end{aligned}$$

∎

Theorem 1.6 can be equally proved as above.

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References

[1] Gerald Folland, *Real Analysis,* 2nd ed. Springer, 2001

Department of Mathematical Sciences, Indiana University - Purdue University Fort Wayne, Fort Wayne, IN 46805-1499

School of Mathematics and Informatics, Jiangxi Normal University, Nanchang, China

E-mail address: `pan@ipfw.edu`