# A Note on Harnack Inequality in the Unit Ball

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## Abstract

A refined estimate of Harnack inequality is proved for positive invariant harmonic functions.

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# A NOTE ON HARNACK INEQUALITY IN THE UNIT BALL

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## 1. INTRODUCTION

Let $B^n = \{x \in \mathbb{R}^n : |x| < 1\},\ n \geq 2$ be the unit ball in $\mathbb{R}^n, S^{n-1} = \partial B^n$. In this paper, we prove a refined estimate of Harnack inequality for positive invariant harmonic functions defined by positive Borel measures on the sphere with respect to the Poisson kernel $P_\lambda$ (defined below).

First we consider positive harmonic functions, and the more general case follows.

**Theorem 1.1.** *Let $u$ be a positive harmonic function in $B^n$. $\xi_1, \xi_2 \in S^{n-1},\ 0 \leq r_1 \leq r_2 < 1$. Then*

$$f(-r_1, -r_2)\exp\{-g(r_1)\} \leq \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \leq f(r_1, r_2)\exp\{g(r_1)\}. \tag{1.1}$$

*where*

$$\begin{aligned} f(r_1, r_2) &= \left(\frac{1+r_2}{1+r_1}\right)\left(\frac{1-r_1}{1-r_2}\right)^{n-1} \\ g(r_1) = g(r_1, \xi_1, \xi_2) &= \frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr_1}{(1-r_1)^2} \end{aligned}$$

**Remark.** When $r_2 = r = |x|, r_1 = 0$, (1.1) becomes

$$\frac{1-r}{(1+r)^{n-1}} \leq \frac{u(x)}{u(0)} \leq \frac{1+r}{(1-r)^{n-1}}$$

— the classical Harnack inequality in $B^n$.

Denote the differential operator

$$\Delta_\lambda = (1-|x|^2)\left\{\frac{1-|x|^2}{4}\sum_j \frac{\partial^2}{\partial x_j^2} + \lambda \sum_j x_j \frac{\partial}{\partial x_j} + \lambda(\frac{n}{2} - 1 - \lambda)\right\},\ \lambda \in \mathbb{R}.$$

Invariant harmonic functions are solutions of $\Delta_\lambda u = 0$ and are of certain invariant property with respect to Möbius transformations. Let $\mu$ be a positive Borel measure on $S^{n-1}$ and $P_\lambda$ be the Poisson kernel

$$P_\lambda = \frac{(1-|x|^2)^{1+2\lambda}}{|x-\eta|^{n+2\lambda}}\ , \quad \lambda \in \mathbb{R}.$$

It is known that

$$u(x) = \int_{S^{n-1}} P_\lambda(x,\eta)d\mu(\eta)$$

is an invariant harmonic function in $B^n$ ([1], p. 119).

**Theorem 1.2.** *Let $u$ be a positive invariant harmonic function in $B^n$ defined by a positive Borel measure $\mu$ on $S^{n-1}$ with the Poisson kernel $P_\lambda$. Let $\xi_1, \xi_2 \in S^{n-1}$ and $0 \le r_1 \le r_2 < 1$.*

*If $\lambda > -\frac{n}{2}$,*

$$f_\lambda(-r_1,-r_2)\exp\{-g_\lambda(r_1)\} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le f_\lambda(r_1,r_2)\exp\{g_\lambda(r_1)\}. \tag{1.2}$$

*If $\lambda < -\frac{n}{2}$,*

$$f_\lambda(r_1,r_2)\exp\{-g_\lambda(r_1)\} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le f_\lambda(-r_1,-r_2)\exp\{g_\lambda(r_1)\} \tag{1.3}$$

*where*

$$\begin{aligned} f_\lambda(r_1,r_2) &= \left(\frac{1+r_2}{1+r_1}\right)^{2\lambda+1}\left(\frac{1-r_1}{1-r_2}\right)^{n-1} \\ g_\lambda(r_1) = g_\lambda(r_1,\xi_1,\xi_2) &= \frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r_1}{(1-r_1)^2} \end{aligned}$$

Case $\lambda = \frac{n}{2}-1$ corresponds to the Laplace-Beltrami operator $\Delta_{\frac{n}{2}-1}$ and the Poincaré metric. It is known ([2]) that, for positive $u$, $\Delta_{\frac{n}{2}-1}u = 0$, there exists a positive Borel measure $\mu$ on $S^{n-1}$ such that

$$u(x) = \int_{S^{n-1}} P_{\frac{n}{2}-1}(x,\eta)d\mu(\eta).$$

In this case, Theorem 1.2 has the following form.

**Corollary 1.3.** *Let $u$ be a positive solution of $\Delta_{\frac{n}{2}-1}u = 0$ in $B^n$. Let $\xi_1, \xi_2 \in S^{n-1}$ and $0 \le r_1 \le r_2 < 1$. Then*

$$\frac{1}{C} \le \frac{u(r_2\xi_2)}{u(r_1\xi_1)} \le C,$$

*where*

$$C = C(r_1,r_2,\xi_1,\xi_2) = \left(\frac{1+r_2}{1+r_1}\cdot\frac{1-r_1}{1-r_2}\right)^{n-1}\exp\left\{\pi|\xi_2-\xi_1|\frac{(n-1)r_1}{(1-r_1)^2}\right\}.$$

The proofs will need a result in [4] on the monotonicity of positive invariant harmonic functions. Here we state the result as a proposition. The proof is provided in [4].

**Proposition 1.4.** *(Theorem 1.2 in* [4]*)*
*Let $u$ be a positive invariant harmonic function defined in $B^n$ by a positive Borel measure $\mu$ on $S^{n-1}$ with the Poisson kernel $P_\lambda$. Let $\zeta \in S^{n-1}$ and $0 \le r' \le r < 1$.*

*If $\lambda > -\frac{n}{2}$,*

$$\left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1} u(r'\zeta).$$

*If $\lambda < -\frac{n}{2}$,*

$$\left(\frac{1+r}{1+r'}\right)^{2\lambda+1}\left(\frac{1-r'}{1-r}\right)^{n-1} u(r'\zeta) \le u(r\zeta) \le \left(\frac{1-r}{1-r'}\right)^{2\lambda+1}\left(\frac{1+r'}{1+r}\right)^{n-1} u(r'\zeta).$$

## 2. Proof of Theorem 1

We need five lemmas before proving Theorem 1.1.

Let $x \cdot y = \sum_{k=1}^{n} x_k y_k$ denote the inner product in $\mathbb{R}^n$.

**Lemma 2.1.** *Let $\eta, \xi_1, \xi_2 \in S^{n-1}$. Let $\varphi(t),\ t \in [0,1]$ be the shortest arc on the great circle connecting $\xi_1$ and $\xi_2$. Then*

$$\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n} = \frac{nr\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2}} \tag{2.1}$$

*Proof.*

$$\frac{d}{dt}|r\varphi(t)-\eta|^2 = \frac{d}{dt}\left(r^2+1-2r\varphi(t)\cdot\eta\right) = -2r\phi'(t)\cdot\eta$$

$$\begin{aligned}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n} &= \frac{d}{dt}\left(|r\varphi(t)-\eta|^2\right)^{-n/2}\\ &= -\frac{n}{2}\left(|r\varphi(t)-\eta|^2\right)^{-\frac{n}{2}-1}\frac{d}{dt}|r\varphi(t)-\eta|^2\\ &= \frac{nr\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2}}\end{aligned}$$

□

**Lemma 2.2.** *Let $u$ be a positive harmonic function in $B^n$. $\xi_1, \xi_2 \in S^{n-1}$. Let $\varphi(t),\ t \in [0,1]$ be the shortest arc on the great circle connecting $\xi_1$ and $\xi_2$. Then*

$$\left|\frac{d}{dt}u(r\varphi(t))\right| \le \frac{nr|\varphi'(t)|}{(1-r)^2}u(r\varphi(t)), \qquad r \in [0,1].$$

*Proof.* By (2.1),

$$\begin{aligned}\int_{S^{n-1}}\left|\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n}\right|d\mu(\eta) &= \int_{S^{n-1}}\left|\frac{nr\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2}}\right|d\mu(\eta)\\ &\le nr\int_{S^{n-1}}\frac{|\varphi'(t)|\ |\eta|}{|r\varphi(t)-\eta|^n(1-r)^2}d\mu(\eta)\\ &= \frac{nr|\varphi'(t)|}{(1-r)^2(1-r^2)}\int_{S^{n-1}}\frac{1-r^2}{|r\varphi(t)-\eta|^n}d\mu(\eta)\\ &= \frac{nr|\varphi'(t)|}{(1-r)^2(1-r^2)}u(r\varphi(t))\end{aligned}$$

where we applied the inequality

$$|r\varphi(t)-\eta| = |1 - r\phi(t)\cdot\eta| \ge 1-r.$$

Therefore by the Lebesgue's Dominant Convergence Theorem,

$$\begin{aligned}\left|\frac{d}{dt}u(r\varphi(t))\right| &= \left|\frac{d}{dt}\int_{S^{n-1}}\frac{1-|r\varphi(t)|^2}{|r\varphi(t)-\eta|^n}d\mu(\eta)\right|\\ &= (1-r^2)\left|\int_{S^{n-1}}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^n}d\mu(\eta)\right|\\ &\le \frac{nr|\varphi'(t)|}{(1-r)^2}u(r\varphi(t)).\end{aligned}$$

□

**Lemma 2.3.** *Let $\xi_1, \xi_2 \in S^{n-1}$. Then there exists a Möbius transformation $T$ in $\mathbb{R}^n$ such that $T(S^{n-1}) = S^{n-1}$ and for all $r \in [0,1]$,*

$$T(r\xi_i) = (0, \cdots, 0, r\cos\theta_i, r\sin\theta_i),\ i = 1, 2,\ |\theta_2 - \theta_1| \le \pi$$

*and*

$$|\det T'(x)| = 1, \quad |T(r\xi_2) - T(r\xi_1)| = |r\xi_2 - r\xi_1|, \tag{2.2}$$

*where $T'(x)$ denotes the Jacobian matrix of the Möbius transformation $T$.*

*Proof.* The transformations involves a rotation in $\mathbb{R}^n$ with respect to the origin such that $\xi_1, \xi_2$ are in the $\mathbb{R}^2$ plane of the last two coordinates. If $|\theta_2 - \theta_1| \le \pi$ is not yet satisfied, it can be achieved by a reflection with respect to the origin. Both rotation and reflection preserve the Euclidean norm and distance, and the absolute value of the determinant of the Jacobian matrix $|\det T'(x)| = 1$. □

**Lemma 2.4.** *Let*

$$\xi_i = (0, \cdots, 0, r\cos\theta_i, r\sin\theta_i) \in S^{n-1},\ i = 1, 2,\ |\theta_2 - \theta_1| \le \pi$$

*and*

$$\varphi(t) = (0, \cdots, 0, r\cos\theta_t, r\sin\theta_t),\ \theta_t = t\theta_2 + (1-t)\theta_1,\ t \in [0,1].$$

*Then*

$$|\varphi'(t)| \le \frac{\pi}{2}|\xi_2 - \xi_1|. \tag{2.3}$$

*Proof.* It suffices to prove for $n = 2$. For computation convenience, we use the complex plane notations in $\mathbb{R}^2$. Denote $\xi_i = e^{1\theta_i}$.

$$\begin{aligned}\xi_2 - \xi_1 &= e^{i\theta_2} - e^{i\theta_1} \\ &= \exp\left(i\frac{\theta_2+\theta_1}{2}\right)\left(\exp\left(i\frac{\theta_2-\theta_1}{2}\right) - \exp\left(-i\frac{\theta_2-\theta_1}{2}\right)\right) \\ &= \exp\left(i\frac{\theta_2+\theta_1}{2}\right)(2i)\sin\frac{\theta_2-\theta_1}{2}\end{aligned}$$

Notice that

$$\left|\sin\frac{x}{2}\right| \ge \left|\frac{x}{\pi}\right| \quad for \quad |x| \le \pi,$$

so $|\theta_2 - \theta_1| \le \pi$ implies

$$\left|e^{i\theta_2} - e^{i\theta_1}\right| = 2\left|\sin\left(\frac{\theta_2-\theta_1}{2}\right)\right| \ge \frac{2}{\pi}|\theta_2 - \theta_2|.$$

Furthermore,

$$\varphi'(t) = \frac{d}{dt}\left(e^{it\theta_2 + i(1-t)\theta_1}\right) = \varphi(t)i(\theta_2 - \theta_1).$$

Thus

$$|\varphi'(t)| = |\theta_2 - \theta_1| \le \frac{\pi}{2}\left|e^{i\theta_2} - e^{i\theta_1}\right| = \frac{\pi}{2}|\xi_2 - \xi_1|.$$

□

**Lemma 2.5.** *Let $u$ be a positive invariant harmonic function. $\xi_1, \xi_2 \in S^{n-1}$. Then for $r \in [0,1)$,*

$$\exp\left\{-\frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2}\right\} \le \frac{u(r\xi_2)}{u(r\xi_1)} \le \exp\left\{\frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2}\right\}.$$

*Proof.* Let $T$ be the Möbius transformation in $\mathbb{R}^n$ such that for $r \in [0,1]$,

$$r\zeta_i = T(r\xi_i) = (0, \cdots, 0, r\cos\theta_i, r\sin\theta_i),\ i = 1, 2,\ |\theta_2 - \theta_1| \le \pi.$$

By (2.2) in Lemma 2.3 and [3],

$$U(x) = u(T^{-1}(x))$$

is still a positive harmonic function in $B^n$ with respect to the measure $\mu(T^{-1}(x))$, $|\det T'(x)| = 1$. Furthermore,

$$U(r\zeta_i) = u(T^{-1}(r\zeta_i)) = u(r\xi_i),\ i = 1, 2$$

and

$$|\xi_2 - \xi_1| = |T(\xi_2) - T(\xi_1)| = |\zeta_2 - \zeta_1| = \left|e^{i\theta_2} - e^{i\theta_1}\right|.$$

Let

$$\varphi(t) = (0, \cdots, 0, r\cos\theta_t, r\sin\theta_t),\ \theta_t = t\theta_2 + (1-t)\theta_1,\ t \in [0, 1]$$

be the shortest arc on the great circle connecting $\zeta_1$ and $\zeta_2$, $\varphi(0) = \zeta_1, \varphi(1) = \zeta_2$. By (2.3) in Lemma 2.4,

$$\begin{aligned}\left|\int_0^1 \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}dt\right| &\leq \int_0^1 \left|\frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}\right| dt \\ &\leq \frac{nr}{(1-r)^2}\int_0^1 |\varphi'(t)|dt \\ &\leq \frac{\pi}{2}|\zeta_2 - \zeta_1|\frac{nr}{(1-r)^2}\end{aligned}$$

Since

$$\ln\frac{u(r\xi_2)}{u(r\xi_1)} = \ln\frac{U(r\zeta_2)}{U(r\zeta_1)} = \ln\frac{U(r\varphi(1))}{U(r\varphi(0))} = \int_0^1 \frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}dt,$$

we have

$$\left|\ln\frac{u(r\xi_2)}{u(r\xi_1)}\right| \leq \frac{\pi}{2}|\zeta_2 - \zeta_1|\frac{nr}{(1-r)^2} = \frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2}.$$

Therefore

$$-\frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2} \leq \ln\frac{u(r\xi_2)}{u(r\xi_1)} \leq \frac{\pi}{2}|\xi_2 - \xi_1|\frac{nr}{(1-r)^2}.$$

This completes the proof of Lemma 2.5. ☐

Now we prove Theorem 1.1.

*Proof.*

$$\frac{u(r_2\xi_2)}{u(r_1\xi_1)} = \frac{u(r_2\xi_2)}{u(r_1\xi_2)}\frac{u(r_1\xi_2)}{u(r_1\xi_1)}$$

Proposition 1.4 implies

$$\frac{1-r_2}{1-r_1}\left(\frac{1+r_1}{1+r_2}\right)^{n-1} \leq \frac{u(r_2\xi_2)}{u(r_1\xi_2)} \leq \frac{1+r_2}{1+r_1}\left(\frac{1-r_1}{1-r_2}\right)^{n-1}.$$

Combine the above with the results in Lemma 2.5. Theorem 1.1 follows. ☐

## 3. Proof of Theorem 1.2

We need the following three lemmas to Theorem 1.2.

**Lemma 3.1.** *Let* $\eta, \xi_1, \xi_2 \in S^{n-1}$*. Let* $\varphi(t),\ t \in [0,1]$ *be the shortest arc on the great circle connecting* $\xi_1$ *and* $\xi_2$*. Then for* $\lambda \in \mathbb{R}$*,*

$$\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}} = \frac{(n+2\lambda)r\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2\lambda+2}} \tag{3.1}$$

*Proof.* The proof is similar to that of Lemma 2.1.

$$\begin{aligned}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}} &= \frac{d}{dt}\left(|r\varphi(t)-\eta|^2\right)^{-\frac{n+2\lambda}{2}}\\ &= -\frac{n+2\lambda}{2}\left(|r\varphi(t)-\eta|^2\right)^{-\frac{n+2\lambda}{2}-1}\frac{d}{dt}|r\varphi(t)-\eta|^2\\ &= \frac{(n+2\lambda)r\varphi'(t)\cdot\eta}{|r\varphi(t)-\eta|^{n+2\lambda+2}}\end{aligned}$$

using the result from the proof of (2.1) in Lemma 2.1. □

**Lemma 3.2.** *Let $u$ be a positive invariant harmonic function in $B^n$ defined by a positive Borel measure $\mu$ on $S^{n-1}$ with the Poisson kernel $P_\lambda$. $\xi_1, \xi_2 \in S^{n-1}$. Let $\varphi(t),\ t \in [0,1]$ be the shortest arc on the great circle connecting $\xi_1$ and $\xi_2$. Then for $\lambda \in \mathbb{R},\ \lambda \neq -\frac{n}{2}$,*

$$\left|\frac{d}{dt}u(r\varphi(t))\right| \le \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2}u(r\varphi(t)), \qquad r \in [0,1]. \tag{3.2}$$

*Proof.* By (3.1) and $|r\varphi(t)-\eta| = |1 - r\phi(t)\cdot\eta| \ge 1-r$,

$$\begin{aligned}\int_{S^{n-1}}\left|\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}}\right|d\mu(\eta) &= \int_{S^{n-1}}\frac{|(n+2\lambda)r\varphi'(t)\cdot\eta|}{|r\varphi(t)-\eta|^{n+2\lambda+2}}d\mu(\eta)\\ &\le |n+2\lambda|r\int_{S^{n-1}}\frac{|\varphi'(t)|\ |\eta|}{|r\varphi(t)-\eta|^{n+2\lambda}(1-r)^2}d\mu(\eta)\\ &= \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2(1-r^2)^{1+2\lambda}}\int_{S^{n-1}}\frac{(1-r^2)^{1+2\lambda}}{|r\varphi(t)-\eta|^{n+2\lambda}}d\mu(\eta)\\ &= \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2(1-r^2)^{1+2\lambda}}u(r\varphi(t)).\end{aligned}$$

By the Lebesgue's Dominant Convergence Theorem,

$$\begin{aligned}\left|\frac{d}{dt}u(r\varphi(t))\right| &= \left|\frac{d}{dt}\int_{S^{n-1}}\frac{(1-|r\varphi(t)|^2)^{1+2\lambda}}{|r\varphi(t)-\eta|^{n+2\lambda}}d\mu(\eta)\right| \\ &= (1-r^2)^{1+2\lambda}\left|\int_{S^{n-1}}\frac{d}{dt}\frac{1}{|r\varphi(t)-\eta|^{n+2\lambda}}d\mu(\eta)\right| \\ &\leq \frac{r|(n+2\lambda)\varphi'(t)|}{(1-r)^2}u(r\varphi(t)).\end{aligned}$$

□

**Lemma 3.3.** *Let $u$ be a positive invariant harmonic function in $B^n$ defined by a positive Borel measure $\mu$ on $S^{n-1}$ with the Poisson kernel $P_\lambda$. $\xi_1, \xi_2 \in S^{n-1}$. Then for $r \in [0,1)$,*

$$\exp\left\{-\frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2}\right\} \leq \frac{u(r\xi_2)}{u(r\xi_1)} \leq \exp\left\{\frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2}\right\} \tag{3.3}$$

*Proof.* Let $T$ be the Möbius transformation in $\mathbb{R}^n$ such that

$$r\zeta_i = T(r\xi_i) = (0,\cdots,0,r\cos\theta_i, r\sin\theta_i),\ i=1,2,\ |\theta_2-\theta_1|\leq\pi.$$

By (2.2) in Lemma 2.3 and [3],

$$U(x) = u(T^{-1}(x))$$

is also a positive invariant harmonic function in $B^n$ with respect to the measure $\mu(T^{-1}(x))$, $|\det T'(x)| = 1$.

$$U(r\zeta_i) = u(T^{-1}(r\zeta_i)) = u(r\xi_i),\ i=1,2$$

and

$$|\xi_2-\xi_1| = |T(\xi_2)-T(\xi_1)| = |\zeta_2-\zeta_1| = |e^{i\theta_2}-e^{i\theta_1}|.$$

Let

$$\varphi(t) = (0,\cdots,0,r\cos\theta_t, r\sin\theta_t),\ \theta_t = t\theta_2+(1-t)\theta_1,\ t\in[0,1].$$

Then $\varphi(0)=\zeta_1, \varphi(1)=\zeta_2$. By (3.2) in Lemma 3.2,

$$\begin{aligned}\left|\int_0^1\frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}dt\right| &\leq \int_0^1\left|\frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}\right|dt \\ &\leq \frac{|n+2\lambda|r}{(1-r)^2}\int_0^1|\varphi'(t)|dt \\ &\leq \frac{\pi}{2}|\zeta_2-\zeta_1|\frac{|n+2\lambda|r}{(1-r)^2}.\end{aligned}$$

Since

$$\ln\frac{u(r\xi_2)}{u(r\xi_1)} = \ln\frac{U(r\zeta_2)}{U(r\zeta_1)} = \ln\frac{U(r\varphi(1))}{U(r\varphi(0))} = \int_0^1\frac{\frac{d}{dt}U(r\varphi(t))}{U(r\varphi(t))}dt,$$

we have

$$\left|\ln\frac{u(r\xi_2)}{u(r\xi_1)}\right| \le \frac{\pi}{2}|\zeta_2-\zeta_1|\frac{|n+2\lambda|r}{(1-r)^2} = \frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2}.$$

Therefore

$$-\frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2} \le \ln\frac{u(r\xi_2)}{u(r\xi_1)} \le \frac{\pi}{2}|\xi_2-\xi_1|\frac{|n+2\lambda|r}{(1-r)^2}.$$

This completes the proof of Lemma 3.3. ☐

The proof Theorem 1.2 is similar to that of Theorem 1.1.

*Proof.*

$$\frac{u(r_2\xi_2)}{u(r_1\xi_1)} = \frac{u(r_2\xi_2)}{u(r_1\xi_2)}\frac{u(r_1\xi_2)}{u(r_1\xi_1)}$$

Apply (3.3) in Lemma 3.3 and Proposition 1.4. Theorem 1.2 follows. ☐

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