

## ICMC Solutions - Spring 2002

1. If  $f(x)$  is continuous on  $[a, b]$ , it is possible for the expression  $\frac{f(x)}{f(x)+f(b-x)}$  to be undefined for all  $x$  in the interval  $[0, b]$ .

(a) Find a nonzero function  $f(x)$  such that  $f(x) + f(b-x) = 0$  for all  $x$  in the interval  $[0, b]$ .

An example of such a function would be  $f(x) = x - \frac{b}{2}$ .

(b) If  $f$  is continuous and positive on  $[0, b]$ , evaluate  $\int_0^b \frac{f(x)}{f(x)+f(b-x)} dx$

$$\begin{aligned} \int_0^b \frac{f(x)}{f(x)+f(b-x)} dx &= \int_0^b \frac{f(x) + f(b-x) - f(b-x)}{f(x)+f(b-x)} dx \\ &= \int_0^b \left( 1 - \frac{f(b-x)}{f(x)+f(b-x)} \right) dx \\ &= b - \int_0^b \frac{f(b-x)}{f(x)+f(b-x)} dx \end{aligned}$$

Now, let  $u = b - x$ . Then,  $du = -dx$ . Also, when  $x = 0$ , we have  $u = b$ , and when  $x = b$ , we have  $u = 0$ . Hence, we have

$$\begin{aligned} \int_0^b \frac{f(x)}{f(x)+f(b-x)} dx &= b - \int_b^0 \frac{f(u)}{f(b-u)+f(u)} (-du) \\ \int_0^b \frac{f(x)}{f(x)+f(b-x)} dx &= b - \int_0^b \frac{f(u)}{f(u)+f(b-u)} du \\ \int_0^b \frac{f(x)}{f(x)+f(b-x)} dx + \int_0^b \frac{f(u)}{f(u)+f(b-u)} du &= b \end{aligned}$$

The variables make no difference in the integrals, so the left-hand side becomes  $2 \int_0^b \frac{f(x)}{f(x)+f(b-x)} dx = b$ , and  $\int_0^b \frac{f(x)}{f(x)+f(b-x)} dx = \frac{b}{2}$

2. Let  $\sum a_n$  be a series of strictly positive terms, and let  $b_n = \frac{a_1+a_2+\dots+a_n}{n}$ . Show  $\sum b_n$  diverges.

Since each  $a_i \geq 0$  for all  $i \geq 1$ , we have  $b_n = \frac{a_1+a_2+\dots+a_n}{n} \geq \frac{a_1}{n} = a_1 \cdot \frac{1}{n}$ . Since the series  $\sum \frac{1}{n}$  diverges, the series  $\sum \frac{a_1}{n}$  diverges, and hence  $\sum b_n$  diverges by the direct comparison test.

3. Show that the sum of two consecutive odd primes has at least three (not necessarily distinct) prime factors.

Let  $p$  and  $q$  be consecutive odd primes with  $p < q$ . Then, we can write  $p = 2k + l$  and  $q = 2k + m$ , where  $l$  and  $m$  are odd numbers with  $l < m$ . Then,  $p + q = 2k + l + 2k + m = 4k + l + m$ . Since  $l$  and  $m$  are both odd, the sum  $l + m$  is even, and hence we can factor 2 out of the right-hand side to get  $p + q = 2 \left( 2k + \frac{l+m}{2} \right)$ . Now, the fraction  $\frac{l+m}{2}$  represents the average of  $l$  and  $m$ , and since  $l < m$ , we have  $l < \frac{l+m}{2} < m$ , and the number  $2k + \frac{l+m}{2}$  is strictly between  $p = 2k + l$  and  $q = 2k + m$ . Since these two primes were consecutive, we must have  $2k + \frac{l+m}{2}$  is composite, and hence it has at least two prime factors. Thus, the sum of two consecutive odd primes has at least three prime factors.

4. A university bookstore sold at least one mathematics textbook each day for 100 consecutive days. During this time, the bookstore sold 140 mathematics textbooks. Was there a period of consecutive days when exactly 59 mathematics textbooks were sold?

Let  $a_n$  denote the total number of books sold on day  $n$ . Hence, we have  $1 \leq a_1 < a_2 < a_3 < \dots < a_{100} = 140$ . Adding 59 to each above term gives  $60 \leq a_1+59 < a_2+59 < a_3+59 < \dots < a_{100}+59 = 199$ . Now, the list  $a_1, a_2, \dots, a_{100}, a_1+59, a_2+59, \dots, a_{100}+59$  is a list of 200 numbers that lie between 1 and 199, inclusive. Hence, there must be two numbers in the list that have the exact same value. Since

$a_1 < a_2 < \dots < a_{100}$ , we know  $a_i \neq a_j$  for any  $i \neq j$ . Similarly, we cannot have  $a_i + 59 = a_j + 59$  for  $i \neq j$ . Hence, we must have  $a_i = a_j + 59$  for some  $i \neq j$ , or  $|a_i - a_j| = 59$ . The quantity  $|a_i - a_j|$  represents the number of books sold between day  $j + 1$  and day  $i$ . Hence, there were exactly 59 books sold from day  $j + 1$  until day  $i$ .

5. Let  $\varphi$  be a function from a set  $S$  with a binary operation (denoted by juxtaposition) to the set of nonnegative integers such that  $\varphi(xy) = \varphi(x)\varphi(y)$  for all  $x$  and  $y$  in  $S$ .

(a) If  $S$  is a group and the binary operation is the group operation, show that either  $\varphi(x) = 0$  for all  $x$  in  $S$ , or  $\varphi(x) = 1$  for all  $x$  in  $S$ .

Let  $e$  be the identity of  $S$ . If there exists an  $x \in S$  such that  $\varphi(x) = 0$ , then

$$\begin{aligned}\varphi(e) &= \varphi(xx^{-1}) \\ &= \varphi(x)\varphi(x^{-1}) \\ &= 0 \cdot \varphi(x^{-1}) \\ &= 0\end{aligned}$$

and for any other  $y \in S$ , we have

$$\begin{aligned}\varphi(y) &= \varphi(ey) \\ &= \varphi(e)\varphi(y) \\ &= 0 \cdot \varphi(y) \\ &= 0\end{aligned}$$

and hence  $\varphi(x) = 0$  for all  $x \in S$ .

Now, suppose  $\varphi(x) \neq 0$  for all  $x \in S$ . Then,

$$\begin{aligned}\varphi(e) &= \varphi(ee) \\ \varphi(e) &= \varphi(e)\varphi(e) \\ \frac{\varphi(e)}{\varphi(e)} &= \varphi(e) \text{ (since } \varphi(e) \neq 0\text{)} \\ \varphi(e) &= 1\end{aligned}$$

and for any  $x \in S$ , we have  $\varphi(e) = \varphi(xx^{-1})$ , or  $1 = \varphi(x)\varphi(x^{-1})$ . Since  $\varphi(x)$  and  $\varphi(x^{-1})$  are nonnegative integers with product 1, we must have  $\varphi(x) = 1$ . Hence,  $\varphi(x) = 1$  for all  $x \in S$ .

(b) If  $\varphi$  is a nonconstant function,  $S$  is a ring, and the binary operation is the ring multiplication, show that  $\varphi(0_S) = 0$ .

If  $\varphi(0_S) \neq 0$ , then for all  $x \in S$ , we get

$$\begin{aligned}\varphi(0_S) &= \varphi(0_S \cdot x) \\ \varphi(0_S) &= \varphi(0_S)\varphi(x) \\ \varphi(x) &= 1\end{aligned}$$

which contradicts  $\varphi$  being nonconstant. Hence, we must have  $\varphi(0_S) = 0$ .

6. Show that a square inscribed in a triangle can enclose at most one-half the area of the triangle.

First, we need the fact that for any numbers  $b$  and  $h$ , we have

$$\begin{aligned}(b-h)^2 &\geq 0 \\ b^2 - 2bh + h^2 &\geq 0 \\ b^2 + h^2 &\geq 2bh\end{aligned}$$

Now, we notice that the square “divides” the triangle into a trapezoid with bases  $b$  and  $s$  and a height of  $s$ , and a triangle with base  $s$  and height  $h - s$ . We must have the sum of these areas equal the total area of the triangle; hence, we have

$$\begin{aligned}
 \frac{1}{2}s(b+s) + \frac{1}{2}s(h-s) &= \frac{1}{2}bh \\
 bs + s^2 + sh - s^2 &= bh \\
 s(b+h) &= bh \\
 s^2(b+h)^2 &= b^2h^2 \\
 s^2(b^2 + 2bh + h^2) &= b^2h^2 \\
 s^2(2bh + 2bh) &\leq b^2h^2 \text{ by above} \\
 s^2(4bh) &\leq b^2h^2 \\
 s^2 &\leq \frac{1}{4}bh = \frac{1}{2} \left( \frac{1}{2}bh \right)
 \end{aligned}$$

Hence, the area of the square is at most one half the area of the triangle.