

SOME NONLINEAR DIFFERENTIAL INEQUALITIES AND AN APPLICATION TO HÖLDER CONTINUOUS ALMOST COMPLEX STRUCTURES

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ABSTRACT. We consider some second order quasilinear partial differential inequalities for real valued functions on the unit ball and find conditions under which there is a lower bound for the supremum of non-negative solutions that do not vanish at the origin. As a consequence, for complex valued functions $f(z)$ satisfying $\partial f/\partial \bar{z} = |f|^\alpha$, $0 < \alpha < 1$, and $f(0) \neq 0$, there is also a lower bound for $\sup |f|$ on the unit disk. For each α , we construct a manifold with an α -Hölder continuous almost complex structure where the Kobayashi-Royden pseudonorm is not upper semicontinuous.

1. INTRODUCTION

We begin with an analysis of a second order quasilinear partial differential inequality for real valued functions of n real variables,

$$(1) \quad \Delta u - B|u|^\varepsilon \geq 0,$$

where $B > 0$ and $\varepsilon \in [0, 1)$ are constants. In Section 2, we use a Comparison Principle argument to show that (1) has “no small solutions,” in the sense that there is a uniform lower bound $M > 0$ for the supremum of solutions u which are nonnegative on the unit ball and nonzero at the origin.

We also consider a generalization of (1):

$$(2) \quad u\Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla}u|^2 \geq 0,$$

and find conditions under which there is a similar property of no small solutions, in Theorem 2.4.

As an application of the results on the inequality (1), we show failure of upper semicontinuity of the Kobayashi-Royden pseudonorm for a family of 4-dimensional manifolds with almost complex structures of regularity $\mathcal{C}^{0,\alpha}$, $0 < \alpha < 1$. This generalizes the $\alpha = \frac{1}{2}$ example of [IPR]; it is known

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([IR]) that the Kobayashi-Royden pseudonorm is upper semicontinuous for almost complex structures with regularity $\mathcal{C}^{1,\alpha}$.

Our construction of the almost complex manifolds in Section 4 is very similar to that of [IPR]; we give the details for the convenience of the reader, and to show how the argument breaks down as $\alpha \rightarrow 1^-$, due to a shrinking radius of the domain. We also take the opportunity in Section 3 to state some Lemmas which allow for a more quantitative description than that of [IPR].

One of the steps in [IPR] is a Maximum Principle argument applied to a complex valued function $h(z)$ satisfying the equation $\partial h / \partial \bar{z} = |h|^{1/2}$, to get the property of no small solutions. The main difference between our paper and [IPR] is the use of a Comparison Principle in Section 2 instead of the Maximum Principle, and we arrive at this result:

Theorem 1.1. *For any $\alpha \in (0, 1)$, suppose $h(z)$ is a continuous complex valued function on the closed unit disk, and on the set $\{z : |z| < 1, h(z) \neq 0\}$, h has continuous partial derivatives and satisfies*

$$(3) \quad \frac{\partial h}{\partial \bar{z}} = |h|^\alpha.$$

If $h(0) \neq 0$ then $\sup |h| > S_\alpha$, where the constant $S_\alpha > 0$ is defined by:

$$(4) \quad S_\alpha = \left(\frac{2(1-\alpha)}{2-\alpha} \right)^{1/(1-\alpha)}.$$

2. SOME DIFFERENTIAL INEQUALITIES

Let D_R denote the open ball in \mathbb{R}^n centered at $\vec{0}$ with radius $R > 0$, and let \bar{D}_R denote the closed ball.

Lemma 2.1. *Given constants $B > 0$ and $0 \leq \varepsilon < 1$, let*

$$M = \left(\frac{B(1-\varepsilon)^2}{2(2\varepsilon + n(1-\varepsilon))} \right)^{\frac{1}{1-\varepsilon}} > 0.$$

Suppose the function $u : \bar{D}_1 \rightarrow \mathbb{R}$ satisfies:

- *u is continuous on \bar{D}_1 ,*
- *$u(\vec{x}) \geq 0$ for $\vec{x} \in D_1$,*
- *on the open set $\omega = \{\vec{x} \in D_1 : u(\vec{x}) \neq 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *for $\vec{x} \in \omega$:*

$$(5) \quad \Delta u(\vec{x}) - B(u(\vec{x}))^\varepsilon \geq 0.$$

If $u(\vec{0}) \neq 0$, then $\sup_{\vec{x} \in D_1} u(\vec{x}) > M$.

Proof. Define a comparison function

$$v(\vec{x}) = M|\vec{x}|^{\frac{2}{1-\varepsilon}},$$

so $v \in \mathcal{C}^2(\mathbb{R}^n)$ since $0 \leq \varepsilon < 1$. By construction of M , it can be checked that v is a solution of this nonlinear Poisson equation on the domain \mathbb{R}^n :

$$\Delta v(\vec{x}) - B|v(\vec{x})|^\varepsilon \equiv 0.$$

Suppose, toward a contradiction, that $u(\vec{x}) \leq M$ for all $\vec{x} \in D_1$. For a point \vec{x}_0 on the boundary of $\omega \subseteq \mathbb{R}^n$, either $|\vec{x}_0| = 1$, in which case by continuity, $u(\vec{x}_0) \leq M = v(\vec{x}_0)$, or $0 < |\vec{x}_0| < 1$ and $u(\vec{x}_0) = 0$, so $u(\vec{x}_0) \leq v(\vec{x}_0)$. Since $u \leq v$ on the boundary of ω , the Comparison Principle ([GT] Theorem 10.1) applies to the subsolution u and the solution v on the domain ω . The relevant hypothesis for the Comparison Principle in this case is that the second term expression of (5), $-BX^\varepsilon$, is weakly decreasing, which uses $B > 0$ and $\varepsilon \geq 0$. (To satisfy this technical condition for all $X \in \mathbb{R}$, we define a function $c : \mathbb{R} \rightarrow \mathbb{R}$ by $c(X) = -BX^\varepsilon$ for $X \geq 0$, and $c(X) = 0$ for $X \leq 0$. Then c is weakly decreasing in X , v satisfies $\Delta v(\vec{x}) + c(v(\vec{x})) \equiv 0$ and u satisfies $\Delta u(\vec{x}) + c(u(\vec{x})) \geq 0$.)

The conclusion of the Comparison Principle is that $u \leq v$ on ω , however $\vec{0} \in \omega$ and $u(\vec{0}) > v(\vec{0})$, a contradiction. \blacksquare

Of course, the constant function $u \equiv 0$ satisfies the inequality (5), and so does the radial comparison function v , so the initial condition $u(\vec{0}) \neq 0$ is necessary.

Example 2.2. In the $n = 1$ case, $M = \left(\frac{B(1-\varepsilon)^2}{2(1+\varepsilon)}\right)^{\frac{1}{1-\varepsilon}}$. For points $c_1, c_2 \in \mathbb{R}$, $c_1 < c_2$, define a function

$$u(x) = \begin{cases} M(x - c_2)^{\frac{2}{1-\varepsilon}} & \text{if } x \geq c_2 \\ 0 & \text{if } c_1 \leq x \leq c_2 \\ M(c_1 - x)^{\frac{2}{1-\varepsilon}} & \text{if } x \leq c_1 \end{cases} .$$

Then $u \in \mathcal{C}^2(\mathbb{R})$, and it is nonnegative and satisfies $u'' = B|u|^\varepsilon$ (the case of equality in the $n = 1$ version of (5)). For $c_1 < 0 < c_2$, this gives an infinite collection of solutions of the ODE $u'' = B|u|^\varepsilon$ which are identically zero in a neighborhood of 0, so the ODE does not have a unique continuation property. For $c_1 > 0$ or $c_2 < 0$, the function u satisfies $u(0) \neq 0$ and the other hypotheses of Lemma 2.1, and its supremum on $(-1, 1)$ exceeds M even though it can be identically zero on an interval not containing 0.

Example 2.3. In the case $n = 2$, $B = 1$, $\varepsilon = 0$, (5) becomes the linear inequality $\Delta u \geq 1$ and the number $M = \frac{1}{4}$ agrees with Lemma 2 of [IPR], which was proved there using a Maximum Principle argument.

By applying Lemma 2.1 to the Laplacian of a power of u , we get the following generalization.

Theorem 2.4. *Given constants $B > 0$, $C \in \mathbb{R}$, and $\varepsilon < 1$, let*

$$M = \begin{cases} \left(\frac{B(1-\varepsilon)^2}{2(2(\varepsilon-C)+n(1-\varepsilon))} \right)^{\frac{1}{1-\varepsilon}} & \text{if } C \leq \varepsilon \\ \left(\frac{B(1-\varepsilon)}{2n} \right)^{\frac{1}{1-\varepsilon}} & \text{if } C \geq \varepsilon \end{cases}.$$

Suppose the function $u : \bar{D}_1 \rightarrow \mathbb{R}$ satisfies:

- u is continuous on \bar{D}_1 ,
- $u(\vec{x}) \geq 0$ for $\vec{x} \in D_1$,
- on the open set $\omega = \{\vec{x} \in D_1 : u(\vec{x}) \neq 0\}$, $u \in \mathcal{C}^2(\omega)$,
- for $\vec{x} \in \omega$:

$$u(\vec{x})\Delta u(\vec{x}) \geq B|u(\vec{x})|^{1+\varepsilon} + C|\vec{\nabla}u(\vec{x})|^2.$$

If $u(\vec{0}) \neq 0$, then $\sup_{\vec{x} \in D_1} u(\vec{x}) > M$.

Proof. Let $\mu = \min\{\varepsilon, C\}$, so $\mu \leq \varepsilon < 1$, and on the set ω ,

$$u(\vec{x})\Delta u(\vec{x}) \geq B|u(\vec{x})|^{1+\varepsilon} + \mu|\vec{\nabla}u(\vec{x})|^2.$$

Consider the function $u^{1-\mu}$ on \bar{D}_1 , so $u^{1-\mu} \in \mathcal{C}^0(\bar{D}_1) \cap \mathcal{C}^2(\omega)$, and on the set ω ,

$$\begin{aligned} \Delta(u^{1-\mu}) &= (1-\mu)u^{-\mu-1}(u\Delta u - \mu|\vec{\nabla}u|^2) \\ &\geq (1-\mu)u^{-\mu-1}Bu^{1+\varepsilon} \\ &= (1-\mu)B(u^{1-\mu})^{(\varepsilon-\mu)/(1-\mu)}. \end{aligned}$$

Since $(1-\mu)B > 0$, and $\mu \leq \varepsilon < 1 \implies 0 \leq \frac{\varepsilon-\mu}{1-\mu} < 1$, Lemma 2.1 applies to $u^{1-\mu}$. If $(u(\vec{0}))^{1-\mu} \neq 0$, then

$$\begin{aligned} \sup u^{1-\mu} &> \left(\frac{(1-\mu)B(1-\frac{\varepsilon-\mu}{1-\mu})^2}{2(2\frac{\varepsilon-\mu}{1-\mu} + n(1-\frac{\varepsilon-\mu}{1-\mu}))} \right)^{\frac{1}{1-\frac{\varepsilon-\mu}{1-\mu}}} \\ \implies \sup u &> \left(\frac{B(1-\varepsilon)^2}{2(2(\varepsilon-\mu) + n(1-\varepsilon))} \right)^{\frac{1}{1-\varepsilon}}. \end{aligned}$$

■

Functions satisfying a differential inequality of the form (1) or (2) also satisfy a Strong Maximum Principle; the only condition is $B > 0$.

Theorem 2.5. *Given any open set $\Omega \subseteq \mathbb{R}^n$, and any constants $B > 0$, $C, \varepsilon \in \mathbb{R}$, suppose the function $u : \Omega \rightarrow \mathbb{R}$ satisfies:*

- *u is continuous on Ω ,*
- *on the set $\omega = \{\vec{x} \in \Omega : u(\vec{x}) > 0\}$, $u \in \mathcal{C}^2(\omega)$,*
- *on the set ω , u satisfies*

$$u\Delta u - B|u|^{1+\varepsilon} - C|\vec{\nabla}u|^2 \geq 0.$$

If $u(\vec{x}_0) > 0$ for some $\vec{x}_0 \in \Omega$, then u does not attain a maximum value on Ω .

Proof. Note that the constant function $u \equiv 0$ is the only locally constant solution of the inequality for $B > 0$. If $B = 0$ then obviously any constant function would be a solution.

Given a function u satisfying the hypotheses, ω is a nonempty open subset of Ω . Suppose, toward a contradiction, that there is some $\vec{x}_1 \in \Omega$ with $u(\vec{x}) \leq u(\vec{x}_1)$ for all $x \in \Omega$. In particular, $u(\vec{x}_1) \geq u(\vec{x}_0) > 0$, so $\vec{x}_1 \in \omega$. Let ω_1 be the connected component of ω containing \vec{x}_1 .

For $\vec{x} \in \omega_1$, u satisfies the linear, uniformly elliptic inequality

$$\Delta u(\vec{x}) + (-B(u(\vec{x}))^{\varepsilon-1})u(\vec{x}) + (-C \frac{\vec{\nabla}u(\vec{x})}{u(\vec{x})}) \cdot \vec{\nabla}u(\vec{x}) \geq 0,$$

where the coefficients (defined in terms of the given u) are locally bounded functions of \vec{x} , and $(-B(u(\vec{x}))^{\varepsilon-1})$ is negative for all $\vec{x} \in \omega$. It follows from the Strong Maximum Principle ([GT] Theorem 3.5) that since u attains a maximum value at \vec{x}_1 , then u is constant on ω_1 . Since the only constant solution is 0, it follows that $u(\vec{x}_1) = 0$, a contradiction. ■

The next Lemma shows how an inequality like (5) with $n = 2$ can arise from a first order PDE for a complex valued function.

Lemma 2.6. *Consider constants $\alpha, \gamma \in \mathbb{R}$ with $0 < \alpha < 1$. Let $\omega \subseteq \mathbb{C}$ be an open set, and suppose $h : \omega \rightarrow \mathbb{C}$ satisfies:*

- *$h \in \mathcal{C}^1(\omega)$,*
- *$h(z) \neq 0$ for all $z \in \omega$,*
- *$\frac{\partial h}{\partial \bar{z}} = |h|^\alpha$ on ω .*

Then, the following inequality is satisfied on ω :

$$(6) \quad \Delta(|h|^{(1-\alpha)\gamma}) \geq (4(1-\alpha)\gamma - (2-\alpha)^2)|h|^{(1-\alpha)(\gamma-2)}.$$

Remark. The special case $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$ is Lemma 1 of [IPR]; its Proof there is a long calculation in polar coordinates, which can be generalized to some other values of α by an analogous argument. However, using z, \bar{z} coordinates allows for a shorter calculation.

Proof of Lemma 2.6. We first want to show that h is smooth on ω , applying the regularity and bootstrapping technique of PDE to the equation $\partial h/\partial \bar{z} = |h|^\alpha$. We recall the following fact (for a more general statement, see Theorem 15.6.2 of [AIM]): for a nonnegative integer ℓ , and $0 < \beta < 1$, if $\varphi \in \mathcal{C}_{loc}^{\ell, \beta}(\omega)$ and $g : \omega \rightarrow \mathbb{C}$ has first derivatives in $L_{loc}^2(\omega)$ and is a solution of $\partial g/\partial \bar{z} = \varphi$, then $g \in \mathcal{C}_{loc}^{\ell+1, \beta}(\omega)$. In our case, $\varphi = |h|^\alpha \in \mathcal{C}^1(\omega) \subseteq \mathcal{C}_{loc}^{0, \beta}(\omega)$ (since $h \in \mathcal{C}^1(\omega)$ and is nonvanishing), and $g = h$ has continuous first derivatives, so we can conclude that $g = h \in \mathcal{C}_{loc}^{1, \beta}(\omega)$. Repeating gives that $h \in \mathcal{C}_{loc}^{2, \beta}(\omega)$, etc.

Since the conclusion is a local statement, it is enough to express ω as a union of open subsets ω_k and establish the conclusion on each subset. For each $z_k \in \omega$, there is a sufficiently small disk ω_k containing z_k , where real exponentiation of $h(z)$ is well-defined on ω_k , by choosing a single-valued branch of log to define $h^r = \exp(r \log(h))$.

The condition $\frac{\partial h}{\partial \bar{z}} = |h|^\alpha$ can be re-written

$$h_{\bar{z}} = (\bar{h})_z = |h|^\alpha = h^{\alpha/2} \bar{h}^{\alpha/2}.$$

This leads to

$$\begin{aligned} h_{z\bar{z}} &= (h_{\bar{z}})_z = (h^{\alpha/2} \bar{h}^{\alpha/2})_z \\ &= \frac{\alpha}{2} (h^{(\alpha/2)-1} \bar{h}^{\alpha/2} h_z + h^\alpha \bar{h}^{\alpha-1}) \\ &= \overline{((\bar{h})_{z\bar{z}})}, \end{aligned}$$

which is used in a line of the next step. For an arbitrary exponent $m \in \mathbb{R}$,

$$\begin{aligned} (|h|^m)_{z\bar{z}} &= (h^{m/2} \bar{h}^{m/2})_{z\bar{z}} \\ &= \frac{\partial}{\partial z} \left(\frac{m}{2} h^{\frac{m}{2}-1} h_{\bar{z}} \bar{h}^{\frac{m}{2}} + h^{\frac{m}{2}} \frac{m}{2} \bar{h}^{\frac{m}{2}-1} (\bar{h})_{\bar{z}} \right) \\ &= \frac{m}{2} \frac{\partial}{\partial z} \left(h^{\frac{m}{2}-1+\frac{\alpha}{2}} \bar{h}^{\frac{m}{2}+\frac{\alpha}{2}} + h^{\frac{m}{2}} \bar{h}^{\frac{m}{2}-1} (\bar{h})_{\bar{z}} \right) \\ &= \frac{m}{2} \left[\left(\frac{m}{2} + \frac{\alpha}{2} - 1 \right) h^{\frac{m}{2}+\frac{\alpha}{2}-2} h_z \bar{h}^{\frac{m}{2}+\frac{\alpha}{2}} \right. \\ &\quad \left. + h^{\frac{m}{2}+\frac{\alpha}{2}-1} \left(\frac{m}{2} + \frac{\alpha}{2} \right) \bar{h}^{\frac{m}{2}+\frac{\alpha}{2}-1} (\bar{h})_z \right. \\ &\quad \left. + \frac{m}{2} h^{\frac{m}{2}-1} h_z \bar{h}^{\frac{m}{2}-1} (\bar{h})_{\bar{z}} \right. \\ &\quad \left. + h^{\frac{m}{2}} \left(\frac{m}{2} - 1 \right) \bar{h}^{\frac{m}{2}-2} (\bar{h})_z (\bar{h})_{\bar{z}} + h^{\frac{m}{2}} \bar{h}^{\frac{m}{2}-1} (\bar{h})_{z\bar{z}} \right]. \\ &= \frac{m}{2} \left[\operatorname{Re}((m + \alpha - 2)|h|^{m+\alpha-4} \bar{h}^2 h_z) + \left(\frac{m}{2} + \alpha \right) |h|^{m+2\alpha-2} \right. \\ &\quad \left. + \frac{m}{2} |h|^{m-2} |h_z|^2 \right]. \end{aligned}$$

With the aim of applying Lemma 2.1 to the function $|h|^m$, we consider the expression (8), with real constants B , ε , and $m \neq 0$. In line (9), we assign

$$(7) \quad \varepsilon = \frac{1}{m}(m + 2\alpha - 2)$$

to be able to combine like terms, and in line (10), we choose $B = 4m - (2 - \alpha)^2$ to complete the square.

$$\begin{aligned}
 (8) \quad & \Delta(|h|^m) - B(|h|^m)^\varepsilon \\
 &= 4(|h|^m)_{z\bar{z}} - B|h|^{m\varepsilon} \\
 &= 2m \left[\operatorname{Re} \left((m + \alpha - 2)|h|^{m+\alpha-4}\bar{h}^2 h_z \right) + \left(\frac{m}{2} + \alpha \right) |h|^{m+2\alpha-2} \right. \\
 &\quad \left. + \frac{m}{2} |h|^{m-2} |h_z|^2 \right] - B|h|^{m\varepsilon} \\
 (9) \quad &= (m(m + 2\alpha) - B)|h|^{m+2\alpha-2} \\
 &\quad + \operatorname{Re} \left(2m(m + \alpha - 2)|h|^{m+\alpha-4}\bar{h}^2 h_z \right) + m^2 |h|^{m-2} |h_z|^2 \\
 &\geq |h|^{m-2} \left((m^2 + 2\alpha m - B)|h|^{2\alpha} \right. \\
 &\quad \left. - 2|m||m + \alpha - 2||h|^\alpha |h_z| + m^2 |h_z|^2 \right) \\
 (10) \quad &= |h|^{m-2} (|m + \alpha - 2||h|^\alpha - |m||h_z|)^2 \geq 0.
 \end{aligned}$$

Considering the form of (7), it is convenient to choose $m = (1 - \alpha)\gamma$ for some constant $\gamma \neq 0$. The claim of the Lemma follows; the $\gamma = 0$ case can be checked separately. \blacksquare

The parameter γ can be chosen arbitrarily large; to apply Lemma 2.1 to get the “no small solutions” result of Theorem 1.1, we need the RHS coefficient in (6) to be positive, so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$, and also the RHS exponent $(1 - \alpha)(\gamma - 2)$ to be nonnegative, so $\gamma \geq 2$. In contrast, the $\alpha = \frac{1}{2}$, $\gamma = \frac{3}{2}$ case appearing in Lemma 1 of [IPR] has RHS exponent $-\frac{1}{4}$. The approach of Theorem 2 of [IPR] is to use the negative exponent together with the result of Example 2.3 to show that assuming h has a small solution leads to a contradiction. As claimed, their method can be generalized to apply to other nonpositive exponents, but $\frac{(2-\alpha)^2}{4(1-\alpha)} < \gamma \leq 2$ holds only for $\alpha < 2(\sqrt{2} - 1) \approx 0.8284$.

Proof of Theorem 1.1. Given a continuous $h : \bar{D}_1 \rightarrow \mathbb{C}$ satisfying the hypotheses of Theorem 1.1, on the set $\omega = \{z \in D_1 : h(z) \neq 0\}$, $h \in \mathcal{C}^1(\omega)$, and the conclusion of Lemma 2.6 can be re-written:

$$(11) \quad \Delta(|h|^{(1-\alpha)\gamma}) \geq (4(1-\alpha)\gamma - (2-\alpha)^2)(|h|^{(1-\alpha)\gamma})^{1-\frac{2}{\gamma}}.$$

The hypotheses of Lemma 2.1 are satisfied with $n = 2$, $u(x, y) = |h(x + iy)|^{(1-\alpha)\gamma}$, and $u(\vec{0}) \neq 0$, when the RHS of (11) has a positive coefficient

(so $\gamma > \frac{(2-\alpha)^2}{4(1-\alpha)}$) and the quantity $\varepsilon = 1 - \frac{2}{\gamma}$ is in $[0, 1)$ (for $\gamma \geq 2$). The conclusion of Lemma 2.1 is:

$$\begin{aligned} \sup_{z \in D_1} |h(z)|^{(1-\alpha)\gamma} &> M = \left(\frac{1}{4} \cdot (4(1-\alpha)\gamma - (2-\alpha)^2) \cdot \left(\frac{2}{\gamma} \right)^2 \right)^{\gamma/2} \\ \implies \sup_{z \in D_1} |h(z)| &> \left(\frac{4(1-\alpha)\gamma - (2-\alpha)^2}{\gamma^2} \right)^{\frac{1}{2(1-\alpha)}}. \end{aligned}$$

We can optimize this lower bound, using elementary calculus to show that the maximum value of $\frac{4(1-\alpha)\gamma - (2-\alpha)^2}{\gamma^2}$ is achieved at the critical point $\gamma = \frac{(2-\alpha)^2}{2(1-\alpha)} > \max \left\{ 2, \frac{(2-\alpha)^2}{4(1-\alpha)} \right\}$, and the lower bound for the sup is S_α as appearing in (4). \blacksquare

Note that S_α is decreasing for $0 < \alpha < 1$, with $S_{1/2} = \frac{4}{9}$, $S_{2/3} = \frac{1}{8}$, and $S_\alpha \rightarrow 0$ as $\alpha \rightarrow 1^-$. This Theorem is used in the Proof of Theorem 4.3.

Example 2.7. As noted by [IPR], a one-dimensional analogue of Equation (3) in Theorem 1.1 is the well-known (for example, [BR] §I.9) ODE $u'(x) = B|u(x)|^\alpha$ for $0 < \alpha < 1$ and $B > 0$, which can be solved explicitly. By an elementary separation of variables calculation, the solution on an interval where $u \neq 0$ is $|u(x)| = (\pm(1-\alpha)(Bx+C))^{\frac{1}{1-\alpha}}$. The general solution on the domain \mathbb{R} is, for $c_1 < c_2$,

$$u(x) = \begin{cases} (1-\alpha)^{\frac{1}{1-\alpha}} (B(x-c_2))^{\frac{1}{1-\alpha}} & \text{if } x \geq c_2 \\ 0 & \text{if } c_1 \leq x \leq c_2 \\ -(1-\alpha)^{\frac{1}{1-\alpha}} (B(c_1-x))^{\frac{1}{1-\alpha}} & \text{if } x \leq c_1 \end{cases}.$$

So $u \in C^1(\mathbb{R})$, and if $u(0) \neq 0$, then $\sup_{-1 < x < 1} |u(x)| > ((1-\alpha)B)^{\frac{1}{1-\alpha}}$.

3. LEMMAS FOR HOLOMORPHIC MAPS

We continue with the D_R notation for the open disk in the complex plane centered at the origin. The following quantitative Lemmas on inverses of holomorphic functions $D_R \rightarrow \mathbb{C}$ are used in a step of the Proof of Theorem 4.3 where we put a map $D_r \rightarrow \mathbb{C}^2$ into a normal form, (14).

Lemma 3.1 ([G] Exercise I.1.). *Suppose $f : D_1 \rightarrow D_1$ is holomorphic, with $f(0) = 0$, $|f'(0)| = \delta > 0$. For any $\eta \in (0, \delta)$, let $s = \left(\frac{\delta-\eta}{1-\eta\delta} \right) \eta$; then the restricted function $f : D_\eta \rightarrow D_1$ takes on each value $w \in D_s$ exactly once. \blacksquare*

The hypotheses imply $\delta \leq 1$ by the Schwarz Lemma.

Lemma 3.2. *For a holomorphic map $Z_1 : D_r \rightarrow D_2$ with $Z_1(0) = 0$, $Z_1'(0) = 1$, if $r > \frac{4\sqrt{2}}{3}$ then there exists a continuous function $\phi : \overline{D}_1 \rightarrow D_r$ which is holomorphic on D_1 and which satisfies $(Z_1 \circ \phi)(z) = z$ for all $z \in \overline{D}_1$.*

Remark. It follows from the Schwarz Lemma that $r \leq 2$, and it follows from the fact that ϕ is an inverse of Z_1 that $\phi(0) = 0$ and $\phi'(0) = 1$.

Proof. Define a new holomorphic function $f : D_1 \rightarrow D_1$ by

$$f(z) = \frac{1}{2} \cdot Z_1(r \cdot z),$$

so $f(0) = 0$, $f'(0) = \frac{r}{2}$, and Lemma 3.1 applies with $\delta = \frac{r}{2}$. If we choose $\eta = \frac{3r}{8}$, then $s = \frac{3r^2}{64-12r^2}$, and the assumption $r > \frac{4\sqrt{2}}{3}$ implies $s > \frac{1}{2}$. It follows from Lemma 3.1 that there exists a function $\psi : D_s \rightarrow D_\eta$ such that $(f \circ \psi)(z) = z$ for all $z \in \overline{D}_{1/2} \subseteq D_s$; this inverse function ψ is holomorphic on $D_{1/2}$. The claimed function $\phi : \overline{D}_1 \rightarrow D_{r\eta} \subseteq D_r$ is defined by $\phi(z) = r \cdot \psi(\frac{1}{2} \cdot z)$, so for $z \in \overline{D}_1$,

$$Z_1(\phi(z)) = Z_1(r \cdot \psi(\frac{1}{2} \cdot z)) = 2 \cdot f(\psi(\frac{1}{2} \cdot z)) = 2 \cdot \frac{1}{2} \cdot z = z. \quad \blacksquare$$

4. J -HOLOMORPHIC DISKS

For $S > 0$, consider the bidisk $\Omega_S = D_2 \times D_S \subseteq \mathbb{C}^2$, as an open subset of \mathbb{R}^4 , with coordinates $\vec{x} = (x_1, y_1, x_2, y_2) = (z_1, z_2)$ and the trivial tangent bundle $T\Omega_S \subseteq T\mathbb{R}^4$. Consider an almost complex structure J on Ω_S given by a complex structure operator on $T_{\vec{x}}\Omega_S$ of the following form:

$$(12) \quad J(\vec{x}) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ \lambda & 0 & 1 & 0 \end{pmatrix},$$

where $\lambda : \Omega_S \rightarrow \mathbb{R}$ is any function.

A differentiable map $Z : D_r \rightarrow \Omega_S$ is a J -holomorphic disk if $dZ \circ J_{std} = J \circ dZ$, where J_{std} is the standard complex structure on $D_r \subseteq \mathbb{C}$. Let $z = x + iy$ be the coordinate on D_r . For J of the form (12), if $Z(z)$ is defined by complex valued component functions,

$$(13) \quad Z : D_r \rightarrow \Omega_S : Z(z) = (Z_1(z), Z_2(z)),$$

then the J -holomorphic property implies that $Z_1 : D_r \rightarrow D_2$ is holomorphic in the standard way.

Example 4.1. If the function $\lambda(z_1, z_2)$ satisfies $\lambda(z_1, 0) = 0$ for all $z_1 \in D_2$, then the map $Z : D_2 \rightarrow \Omega_S : Z(z) = (z, 0)$ is a J -holomorphic disk.

Definition 4.2. The Kobayashi-Royden pseudonorm on Ω_S is a function $T\Omega_S \rightarrow \mathbb{R} : (\vec{x}, \vec{v}) \mapsto \|(\vec{x}, \vec{v})\|_K$, defined on tangent vectors $\vec{v} \in T_{\vec{x}}\Omega_S$ to be the number

$$\text{glb} \left\{ \frac{1}{r} : \exists \text{ a } J\text{-holomorphic } Z : D_r \rightarrow \Omega_S, Z(0) = \vec{x}, dZ(0)\left(\frac{\partial}{\partial x}\right) = \vec{v} \right\}.$$

Under the assumption that $\lambda \in \mathcal{C}^{0,\alpha}(\Omega_S)$, $0 < \alpha < 1$, it is shown by [IR] and [NW] that there is a nonempty set of J -holomorphic disks through \vec{x} with tangent vector \vec{v} as in the Definition, so the pseudonorm is a well-defined function. Further, each such disk satisfies $Z \in \mathcal{C}^1(D_r)$.

At this point we pick $\alpha \in (0, 1)$ and set $\lambda(z_1, z_2) = -2|z_2|^\alpha$. Let $S = S_\alpha > 0$ be the constant defined by formula (4) from Theorem 1.1. Then, (Ω_S, J) is an almost complex manifold with the following property:

Theorem 4.3. *If $0 \neq b \in D_S$ then $\|(0, b), (1, 0)\|_K \geq \frac{3}{4\sqrt{2}}$.*

Remark. Since $\frac{3}{4\sqrt{2}} \approx 0.53$, and $\|(0, 0), (1, 0)\|_K \leq \frac{1}{2}$ by Example 4.1, the Theorem shows that the Kobayashi-Royden pseudonorm is not upper semi-continuous on $T\Omega_S$.

Proof. Consider a J -holomorphic map $Z : D_r \rightarrow \Omega_S$ of the form (13), and suppose $Z(0) = (0, b) \in \Omega_S$ and $dZ(0)\left(\frac{\partial}{\partial x}\right) = (1, 0)$. Then the holomorphic function $Z_1 : D_r \rightarrow D_2$ satisfies $Z_1(0) = 0$, $Z_1'(0) = 1$, and $Z_2 \in \mathcal{C}^1(D_r)$ satisfies $Z_2(0) = b$.

Suppose, toward a contradiction, that there exists such a map Z with $b \neq 0$ and $r > \frac{4\sqrt{2}}{3}$. Then Lemma 3.2 applies to Z_1 : there is a reparametrization ϕ which puts Z into the following normal form:

$$(14) \quad \begin{aligned} (Z \circ \phi) : \overline{D}_1 &\rightarrow \Omega_S \\ z &\mapsto (Z_1(\phi(z)), Z_2(\phi(z))) = (z, f(z)), \end{aligned}$$

where $f = Z_2 \circ \phi : \overline{D}_1 \rightarrow D_S$ satisfies $f \in \mathcal{C}^0(\overline{D}_1) \cap \mathcal{C}^1(D_1)$. From the fact that $Z \circ \phi$ is J -holomorphic on D_1 , it follows from the form (12) of J that if $f(z) = u(x, y) + iv(x, y)$, then f satisfies this system of nonlinear Cauchy-Riemann equations on D_1 :

$$(15) \quad \frac{du}{dy} = -\frac{dv}{dx} \quad \text{and} \quad \frac{du}{dx} + \lambda(z, f(z)) = \frac{dv}{dy}$$

with the initial conditions $f(0) = b$, $u_x(0) = u_y(0) = v_x(0) = 0$ and $v_y(0) = \lambda(0, b) = -2|b|^\alpha$. The system of equations implies

$$\begin{aligned}
 \frac{\partial f}{\partial \bar{z}} &= \frac{1}{2} \left(\frac{\partial}{\partial x} (u + iv) + i \frac{\partial}{\partial y} (u + iv) \right) \\
 &= \frac{1}{2} (u_x - v_y + i(v_x + u_y)) \\
 (16) \qquad &= -\frac{1}{2} \lambda(z, f(z)) = |f|^\alpha.
 \end{aligned}$$

So, Theorem 1.1 applies, with $f = h$. The conclusion is that

$$\sup_{z \in D_1} |f(z)| > S_\alpha,$$

but this contradicts $|f(z)| < S = S_\alpha$. ■

The previously mentioned existence theory for J -holomorphic disks shows there are interesting solutions of the equation (16), and therefore also the inequality (11).

Example 4.4. For $0 < \alpha < 1$, (Ω_S, J) , $\lambda(z_1, z_2) = -2|z_2|^\alpha$ as above, a map $Z : D_r \rightarrow \Omega_S$ of the form $Z(z) = (z, f(z))$ is J -holomorphic if $f(x, y) = u(x, y) + iv(x, y)$ is a solution of (15). Again generalizing the $\alpha = \frac{1}{2}$ case of [IPR], examples of such solutions can be constructed (for small r) by assuming $v \equiv 0$ and u depends only on x , so (15) becomes the ODE $u'(x) - 2|u(x)|^\alpha = 0$. This is the equation from Example 2.7; we can conclude that J -holomorphic disks in Ω_S do not have a unique continuation property.

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